California State Science Fair

Introduction to Space Flight Mission Design
Part 1 -- Orbital Mechanics

Edward Ruth
drruth@ix.netcom.com

1) Introduction

We don't get nearly as many science fair projects on space flight as I would like to see. I think that one reason is that students think that space flight is too hard for a science fair project. While it is true that most of us can't get an experiment on the Space Shuttle I think that there are still some good science fair projects that have never been done and are just waiting for the right students. The goal of this series of lessons is to give students an introduction to astronautics. You are going to learn something about space operations and you are going to get some tools that you can use to plan your own space missions! Hopefully some of this will stimulate you to think about what you might want to do for a science fair project.

This lesson is on orbit design. We will learn enough about basic orbital mechanics to plan a simple space mission. We see how how to select the most efficient orbit to minimize the amount of rocket propellant required to complete the mission, how to calculate how much propellant will be used, and how long it will take to complete the mission. So strap in, count down, and blast off into the Final Frontier!

2) Basic Orbital Mechanics

I suppose almost everyone today understands that planets, satellites (both natural ones like our Moon and artificial ones like the Mir space station), asteroids, comets, etc. are all in orbit about a central body. The Earth orbits the sun and the Moon and Mir orbit the Earth. But not that long ago the motion of heavenly bodies was a big mystery. One of the great intellectual achievements of all time was when Isaac Newton was able to use the calculus to calculate the shape of the paths of planets and satellites about their central bodies. Newton showed these paths lie in single plane and are what mathematicians call *conic sections*. This means we can use the equation for conic sections to calculate the shapes of our orbits. (They are called conic sections because these curves are the exposed surface you get by cutting through a solid cone at some angle. Now why someone would want to cut a solid cone at some angle is beyond me. I guess in the old days mathematicians had a lot of time on their hands to think of these things.)

The easiest way to write down the equation for a conic section is to use polar coordinates. Most of you are probably familiar with using Cartesian coordinates to plot the location of a point. In a Cartesian coordinate system the location of a point in a plane is given by two numbers: the x-coordinate and the y-coordinate. In a polar coordinate system we still need only two numbers but this time we will use the distance of the point from the origin (*radius* or r*)* and the angle formed by a line drawn from the origin to the point and the x-axis (*azimuth* or θ). We can convert from Cartesian to polar coordinates using the following formulas:

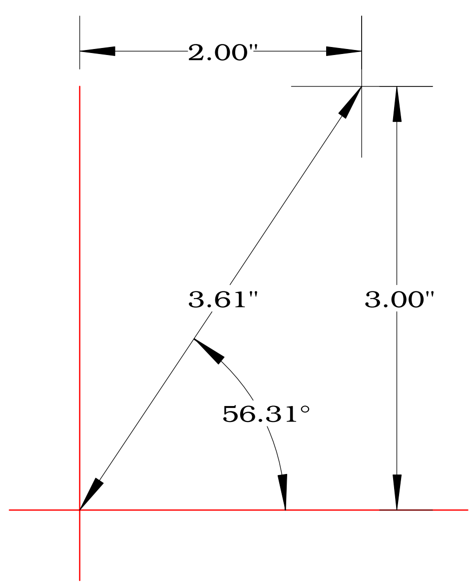
$$x(r,\theta) := r\cdot\cos(\theta) \qquad \text{and} \qquad y(r,\theta) := r\cdot\sin(\theta)$$

$$r(x,y) := \sqrt{x^2 + y^2} \qquad \text{and} \qquad \theta(x,y) := \operatorname{atan}\left(\frac{y}{x}\right)$$

(If you have not had any trigonometry yet you may not be familiar with the sine, cosine, and arc tangent functions used here and later in the lesson. I don't have space here to explain all of trigonometry but if you don't know what these functions are don't let that stop you from reading on. Try to find someone to help you with trig functions. If you still have questions just send me an e-mail and I will try to explain them more fully.)

Let's use these formulas to examine how to plot a point using both Cartesian and polar coordinates:

Fig. 1 -- Polar and Cartesian Coordinates



Given $x_1 := 2\cdot in \qquad y_1 := 3\cdot in$

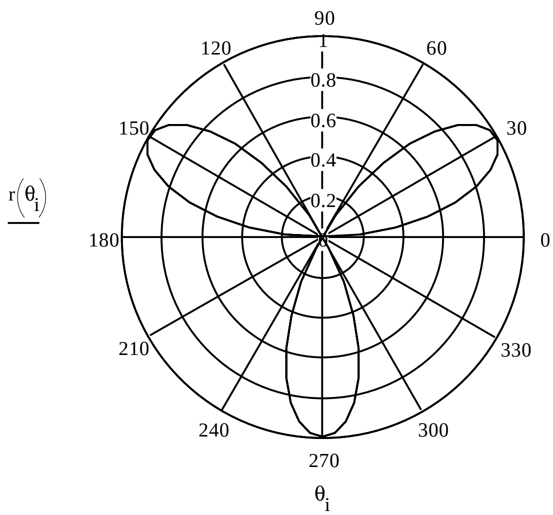
then $r(x_1, y_1) = 3.61\cdot in$

$\theta(x_1, y_1) = 56.31\cdot deg$

Using equations written in polar coordinates we can get some pretty interesting graphs. For example let's look at the "Rose Curve" using polar coordinates:

$N := 3 \qquad r(\theta) := \sin(N\cdot\theta) \qquad i := 1 .. 100 \qquad \theta_i := i\cdot\frac{2\cdot\pi}{100}$

Fig. 2 -- Rose Curve



Now that we have a feel for polar coordinates we can get back to orbital mechanics. As we said the general equation for all orbits is the one for conic sections. This equation is:

$$r(\theta, e) := \frac{H}{1 + e \cdot \cos(\theta)}$$

The parameter e is know as the eccentricity. The value of this parameter defines the shape of our orbit. Depending on the value of e there are four kinds of conic sections which means there are four kinds of orbits:

If $e < 1$ we have the conic section known as the ellipse. The orbit is closed.

A special case of the ellipse with $e = 0$ is the circle.

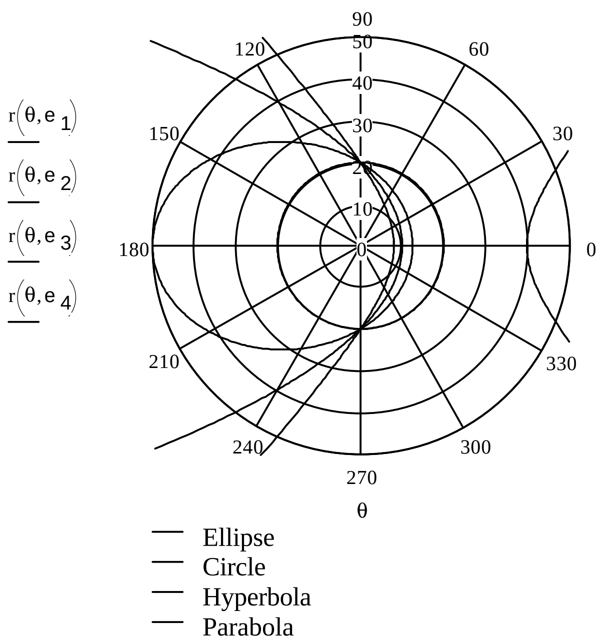
If $e > 1$ we have the conic section known as the hyperbola. The orbit will be open.

A special case of the hyperbola with $e = 1$ is the parabola.

As an example use the following values to plot some conic sections:

$H \equiv 20 \qquad e_1 := .6 \qquad e_2 := 0 \qquad e_3 := 1.5 \qquad e_4 := 1 \qquad \theta := 0, .01 .. 2.05 \cdot \pi$

Fig. 3 -- Plot of Conic Sections

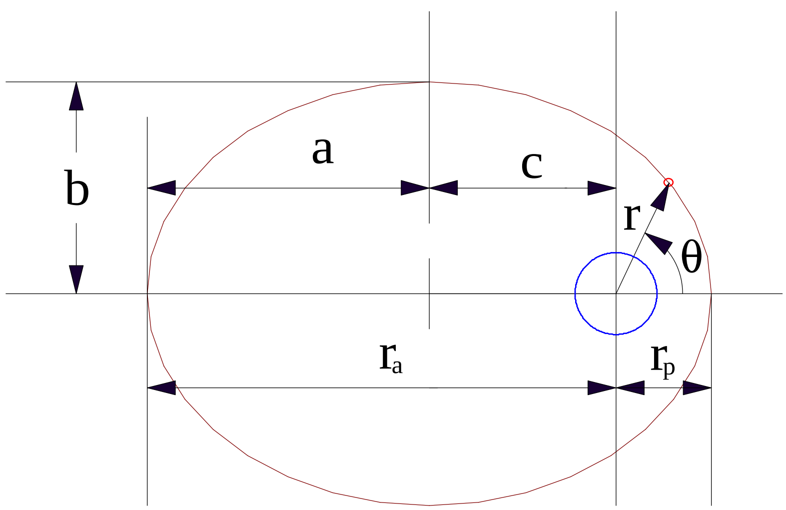


Elliptical orbits are closed orbits. In a closed orbit the satellite will continue to follow the same path forever (neglecting any outside force acting on the satellite such as aerodynamic drag or the gravitational pull from another body). Hyperbolic orbits are open. These are escape orbits. An object in a hyperbolic orbit has enough energy to escape the gravitational pull of the central body and go off into space. The Voyager and Pioneer spacecraft which are leaving our solar system for interstellar space are in hyperbolic orbits about the Sun.

For this lesson we will be concentrating on elliptical orbits. Hyperbolic orbits are only important if you are planning a mission that goes from one central body to another and you want to calculate the escape path. (For example in a mission to Mars the spacecraft must first blast into a hyperbolic orbit about the Earth. After escaping the Earth's gravity the spacecraft will be in an elliptical orbit about the Sun on its trip to Mars.) The most common space missions these days are Earth orbital missions and to understand these we are going to need to know elliptical orbits inside and out.

Figure 4 show us the parameters that we use to define an elliptical orbit. The parameter a is the *semimajor axis* and b is the *semiminor axis*. The point of closest approach to the central body is denoted r_p and is called the *periapsis radius* (if the central body is the Earth then it is called the *perigee radius*. Sometimes I think they make up these names just to keep us confused.). The farthest point is the *apoapsis radius* (*apogee radius* for the Earth) and we use r_a for this parameter.

Fig. 4 -- Elliptical Orbit Parameters



The equation for an elliptical orbit is given by:

$$r(\theta) = \frac{a \cdot \left(1 - e^2\right)}{1 + e \cdot \cos(\theta)}$$

You will find the following relations to be very useful:

$$e = \frac{c}{a} \qquad e = \frac{r_a - r_p}{r_a + r_p}$$

$$a = \frac{r_a + r_p}{2} \qquad a^2 = b^2 + c^2$$

$$r_a = a \cdot (1 + e) \qquad r_p = a \cdot (1 - e)$$

Suppose we wanted to design an elliptical orbit for a spacecraft so that its lowest altitude will be 200 km above the Earth's surface and its highest altitude it will be at 5000 km. First we will find the perigee and apogee. Remember that the radius is measured from the center of the body not from the surface so we have to add the radius of the Earth to the altitudes to get perigee and apogee.

The radius of the Earth is: $R := 6.378 \cdot 10^3 \cdot km$

So we have: $h_p := 200 \cdot km$ $\quad r_p := h_p + R$ $\quad r_p = 6.578 \cdot 10^3 \cdot km$

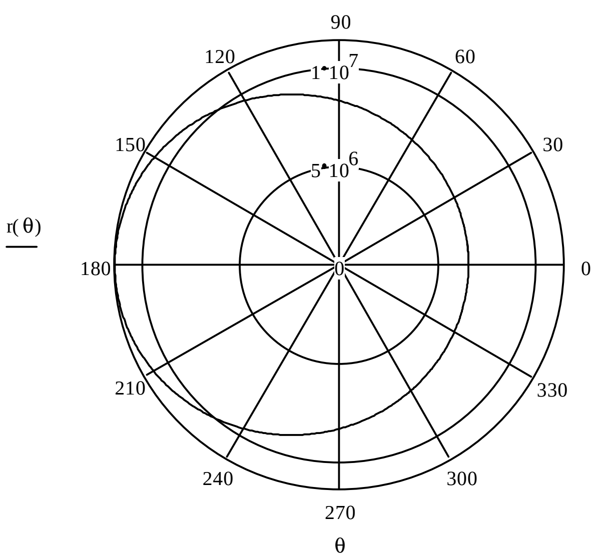
and: $h_a := 5000 \cdot km$ $\quad r_a := h_a + R$ $\quad r_a = 1.138 \cdot 10^4 \cdot km$

Next we find the semimajor axis: $a := \dfrac{r_a + r_p}{2}$ $\quad a = 8.978 \cdot 10^3 \cdot km$

and the eccentricity: $e := \dfrac{r_a - r_p}{r_a + r_p}$ $\quad e = 0.267$

$$r(\theta) := \frac{a \cdot (1 - e^2)}{1 + e \cdot \cos(\theta)}$$

Fig. 5 -- Elliptical Orbit



Try experimenting with different values of the parameters to see the effect on the shape of the orbit. For example set h_a = 50 000 km.

The time it takes our spacecraft to complete a single orbit is called the *period* of the orbit. The period is given by the following expression:

The period => $\tau := 2\cdot\pi\cdot\sqrt{\frac{a^3}{G\cdot M}}$ $\tau = 141.026\cdot min$

where we define the following physical constants:

Gravitational constant => $G \equiv 6.67259\cdot 10^{-11}\cdot\frac{m^3}{kg\cdot sec^2}$

Mass of the central body (for this case the Earth) => $M \equiv 5.98\cdot 10^{24}\cdot kg$

The velocity at periapsis is given by: $v_p := \sqrt{\frac{G\cdot M}{r_p}}\cdot\left(\sqrt{\frac{2\cdot r_a}{r_a + r_p}}\right)$ $v_p = 8.768\cdot\frac{km}{sec}$

The velocity at apoapsis is given by: $v_a := \sqrt{\frac{G\cdot M}{r_a}}\cdot\left(\sqrt{\frac{2\cdot r_p}{r_a + r_p}}\right)$ $v_a = 5.069\cdot\frac{km}{sec}$

And another useful formula is: $r_p\cdot v_p = r_a\cdot v_a$

So now we know how to calculate the shape and period of orbits. The last thing we need to know is how to plan a maneuver from one orbit to another. A very common practice in astronautics is that the launch vehicle will place the payload into a low-altitude, circular, *parking orbit*. The spacecraft will then be boosted from this parking orbit to its operational orbit (which is also often circular). To get from one orbit to the next the mission planner must design a *transfer orbit*. Walter Hohmann showed in 1925 that the most fuel efficient way to do this is to design your transfer orbit so that it is tangent to the initial and final orbits. This kind of orbit is now called a *Hohmann transfer*.

To blast out of the parking orbit to the transfer orbit we need to make a velocity change called a *delta-V* or ΔV. (The Greek letter Δ is used to denote a change. ΔV means a change in velocity.) When the spacecraft arrives at the final orbit we need to make another ΔV to bring us to the velocity needed to orbit at that altitude. In between, during the transfer orbit, the spacecraft just coasts. Calculating the total delta-V required to reach the final destination is the key to mission planning. Once you know the total delta-V then you know how much propellant you need and what size your payload can be. If the mission requires a large delta-V than we are going to have to carry more propellant or make due with a smaller payload. Using the Hohmann transfer minimizes the total delta-V required to complete the mission.

Let's look at an example problem of how to design a Hohmann transfer: An engineer at a communications satellite company needs to size the upper stage needed to place a new spacecraft into a 24 hr equatorial *Clarke orbit*. Clarke orbits (named after Arthur C. Clarke. They are also called *geostationary* or *geosynchronous* orbits) are used for communication satellites because their 24 hr periods match the rotation of the Earth so that the satellite remains stationary over the same location on the ground.

She knows that the launch vehicle will deliver the payload into a circular parking orbit at an altitude of 200 km.

So she has for the parking orbit: $h_1 := 200 \cdot km \quad r_1 := h_1 + R \qquad r_1 = 6.578 \cdot 10^3 \cdot km$

The radius of the final orbit is: $r_2 := \left[\left(\frac{24 \cdot hr}{2 \cdot \pi}\right)^2 \cdot G \cdot M\right]^{\frac{1}{3}} \qquad r_2 = 4.226 \cdot 10^4 \cdot km$

The transfer ellipse needs to be tangent to the parking orbit at perigee and tangent to the final orbit at apogee. This gives her the shape of the parking orbit.

At perigee the transfer orbit must have: $r_p := r_1$

At apogee the transfer orbit must have: $r_a := r_2$

The eccentricity of the transfer orbit is: $e := \frac{r_2 - r_1}{r_2 + r_1} \qquad e = 0.267$

The delta-V required from the first burn is given by:

$$\Delta v_1 := \sqrt{\frac{G \cdot M}{r_1}} \cdot \left(\sqrt{\frac{2 \cdot r_2}{r_2 + r_1}} - 1 \right) \qquad \Delta v_1 = 2.457 \cdot \frac{km}{sec}$$

The delta-V required from the second burn is given by:

$$\Delta v_2 := \sqrt{\frac{G \cdot M}{r_2}} \cdot \left(1 - \sqrt{\frac{2 \cdot r_1}{r_2 + r_1}} \right) \qquad \Delta v_2 = 1.478 \cdot \frac{km}{sec}$$

The total delta-V required for the mission is: $\Delta v := \Delta v_1 + \Delta v_2 \qquad \Delta v = 3.935 \cdot \frac{km}{sec}$

She knows her payload mass is: $m_{payload} := 3000 \cdot kg$

How much propellant is required for this delta-V and mass? The engineer calculates this using the *Rocket Equation*.

$$I_{sp} := 350 \cdot sec \qquad \mathfrak{m} := \exp\left(\frac{\Delta v}{g \cdot I_{sp}} \right) \qquad \mathfrak{m} = 3.147$$

The parameter I_{sp} (pronounced eye-s-pee) is the *specific impulse* and is a measure of the efficiency of the propulsion system. For practical chemical rockets I_{sp} is between 290 and 400 seconds and is mostly a function of the propellant combination used. $\mathfrak{m}$ is the initial to final mass ratio. The propellant required for the mission is thus calculated to be:

$$m_{propellant} := m_{payload} \cdot (\mathfrak{m} - 1) \qquad m_{propellant} = 6.442 \cdot 10^3 \cdot kg$$

How long is the coast to the final orbit? Since we are in the transfer orbit for exactly one half of the ellipse then the duration of the coast is just one half of the period of the complete ellipse:

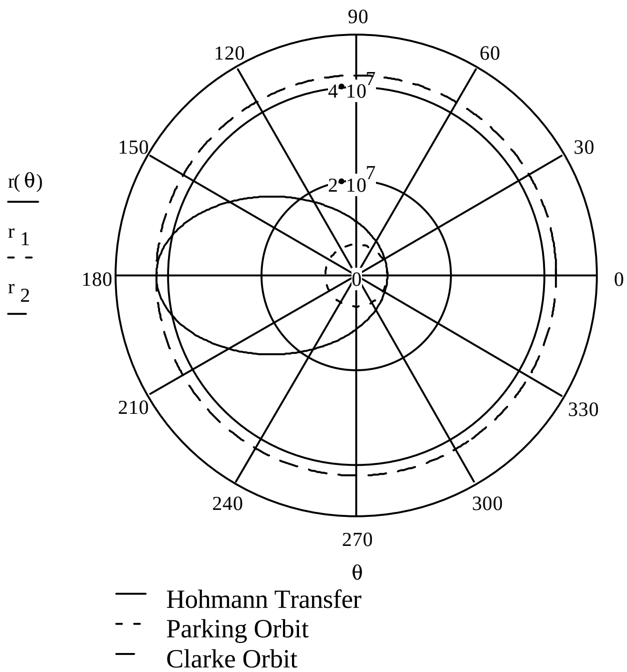
$$a := \frac{r_1}{(1 - e)} \qquad \tau := 2 \cdot \pi \cdot \sqrt{\frac{a^3}{G \cdot M}}$$

$$\tau = 632.512 \cdot min \qquad \frac{\tau}{2} = 316.256 \cdot min$$

This completes the mission design. We can now plot the parking, transfer, and final orbits.

$$r(\theta) := \frac{r_1 \cdot (1 + e)}{1 + e \cdot \cos(\theta)}$$

Fig. 6 -- Hohmann Transfer Orbit



For the next example we will look at an *aerocapture* maneuver. In an aerocapture the spacecraft uses the atmosphere of a planet to slow it down instead of firing its rockets. The spacecraft makes a close pass through the top of the atmosphere in a *grazing trajectory*. Atmospheric drag slows it down providing a ΔV without a burn. The trade off is that although the aerocapture saves propellant you have to carry a *heat shield* to withstand the *aeroheating* caused by passage through the atmosphere. This heat shield can be quite massive. If the heat shield masses more than the propellant you would have saved then there is no point in using aerobraking!

Our next problem will be to calculate the break-even point for a heat shield for a spacecraft entering orbit about Mars. The spacecraft is in a hyperbolic trajectory with respect to the Mars. It is desired to place the spacecraft into a circular parking orbit. What is the mass of propellant required if rocket propulsion is used instead of aerocapture?

The total ΔV required for this maneuver is: $\Delta v := 13.7 \cdot \frac{km}{sec}$

The spacecraft propulsion system has an I_{sp} of: $I_{sp} := 410 \cdot sec$

The mass ratio is found using the rocket equation: $\mathfrak{M} := \exp\left(\frac{\Delta v}{g \cdot I_{sp}}\right)$ $\mathfrak{M} = 30.185$

The dry mass of the spacecraft is: $m_{payload} := 1000 \cdot kg$

The required propellant mass is =>

$m_{propellant} := m_{payload} \cdot (\mathfrak{M} - 1)$ $m_{propellant} = 2.918 \cdot 10^4 \cdot kg$

That's a lot of propellant. The heat shield looks like a reasonable approach for this mission.

Our last example is the case of the crippled bird limping into its final orbit. A spacecraft was successfully launched and injected into a Hohmann transfer to Clarke orbit. However the upper stage failed to fire for the second delta-V burn. Instead of going into Clarke orbit it remains in the Hohmann transfer ellipse. This is the kind of thing that astronautical engineers have nightmares about (well, maybe the other guys say they don't but I sure have them!).

All attempts to get the upper stage to fire have failed and a command is sent to separate it from the spacecraft. The only way to salvage something from the mission is to use the spacecraft's on-board station keeping thrusters to make the injection burn into Clarke orbit. The question is: does the spacecraft have enough on-board propellant to make the burn?

From the first example we know: $\Delta v_2 = 1.478 \cdot \frac{km}{sec}$

The spacecraft has a *bipropellant* thruster system with an I_{sp} of: $I_{sp} := 290 \cdot sec$

The mass ratio is then: $\mathcal{M} := \exp\left(\frac{\Delta v_2}{g \cdot I_{sp}}\right)$ $\mathcal{M} = 1.682$

The dry mass of the spacecraft is: $m_{payload} := 2000 \cdot kg$

The total mass of station keeping propellant is: $m_{propellant\_total} := 1500 \cdot kg$

The required propellant mass is =>

$$m_{propellant\_used} := \left(m_{payload} + m_{propellant\_total}\right) \cdot \left(\frac{\mathcal{M} - 1}{\mathcal{M}}\right)$$

$$m_{propellant\_used} = 1.419 \cdot 10^3 \cdot kg$$

They can just make it! But clearly they face a serious loss of operational lifetime using so much propellant so early in the mission. If they are careful they can husband their gas so that the satellite will last until they can launch a replacement bird.

Suggested Problems

1) The Hohmann transfer uses two burns. Can you devise a transfer ellipse to go from one circular orbit to another using *three* burns?

2) A space vehicle is in a 500 km altitude circular orbit about the Earth. The crew wants to go into a 350 km altitude circular orbit. What is the delta-V required for a Hohmann transfer? For an I_{sp} of 450 sec what is the mass ratio?

3) The spacecraft of the Global Positioning System (GPS) are in 12 hour circular orbits. At what altitude above the Earth are these satellites?

4) The Clarke equatorial orbit does not provide good communication satellite coverage for countries in northern latitudes. The solution devised by Russian astronautical engineers is the *Molniya* orbit. The Molniya orbit is an inclined elliptical orbit in which the spacecraft spends most of its time at high altitude providing good ground coverage for 8 hour periods. A constellation of three satellites will provide continuous coverage. A Molniya orbit requires the following orbital properties:

$$\tau := 43082 \cdot \text{sec}$$

$$a := 26562 \cdot \text{km}$$

Any combination of r_a and r_p for these values of a and τ will work. The easiest Molniya orbit design has a perigee radius equal to the radius of the parking orbit. If the parking orbit altitude is 623 km what is the ΔV required for this orbit? For an I_{sp} of 300 sec what is the mass ratio?

Suggested Further Reading

1) Damon, T. D., *Introduction to Space: the Science of Spaceflight*, Krieger Publishing Company, 1989. (This is a good book for high school students at all levels. It is maybe too light on math for advanced students but covers a lot of material.)

2) Brown, C. D., *Spacecraft Mission Design*, American Institute of Aeronautics and Astronautics, 1992. (Although this book is intended for college students it avoids higher math and is easy to read. It comes with neat free software for doing orbital calculations.)

3) Hale, F. J., *Introduction to Space Flight*, Prentice Hall, 1994. (Much heavier going than the other two and only for advanced high school students. Still it is the best and most complete introductory level text I've seen.)