EXTENDED HASHIN-SHTRIKMAN VARIATIONAL PRINCIPLES

P. Procházka, and J. Sejnoha

Czech Technical University, CZ-166 29 Prague, Czech Republic

sejnohaj@fsv.cvut.cz, *petrp@fsv.cvut.cz*

Abstract

Eigenstresses and eigenstrains act out a very important role in many branches of applied mechanics, e. g. in composites, geotechnics, concrete etc. In previous papers, Procházka (1995), and Procházka, and Šejnoha (1996), have formulated an effective approach for the analysis of homogenization and optimization of nonhomogeneous bodies with prescribed boundary displacements, or tractions. They have used the transformation field analysis, Dvorak, and Procházka (1996), and Dvorak, Procházka, and Shrinivas (1999), for relating the components of stress or strain tensors and the components of eigenstrains or eigenstresses. The approach was based on the idea of combination of augmented Hashin-Shtrikman variational principles and the boundary element method. This paper deals with extended primary and dual variational principles for nonhomogeneous bodies including random properties in the microstructure level. By means of internal parameters, eigenstrains or eigenstresses, involved in H-S principles, it is possible to estimate new bounds on mechanical properties of trial material, increase the bearing capacity of structures, and to minimize the stress excesses.

Introduction

The internal parameters, eigenstrains, or eigenstresses, arise in functionally graded materials, which are typically present in particulate, layered, or rock bodies. These parameters may be realized by means of different sources, e.g. by prestressing, by a change of temperature, by effects of wetting, swelling, they may stand for plastic strains, etc.

In applied mechanics, the extended variational principles involving the eigenparameters that can solve in a non-traditional way many kinds of mechanical problems, posses a particular significance. One of those such possibilities is described in this paper.

The solution of problems involving linear and nonlinear behavior of nonhomogeneous and anisotropic bodies is frequently formulated in terms of variational principles. The natural way to solve these problems is to start with an integral formulation of cost functional and describe the mechanical behavior of the bodies under external and internal loading by searching for its stationary point. The well-known Hashin-Shtrikman variational principles, Hashin-Shtrikman (1962), (1992) are extended and involve eigenstrains or eigenstresses. The surface displacements, or tractions are prescribed along the entire, or split to a parts of the boundary and no volume forces are considered. In order to make clear the use of eigenparameters (eigenstrains and eigenstresses) in physical formulations, the classical .formulation of elasticity is presented. Then Hashin-Shtrikman principles with the influence of eigenparameters are briefly derived. Reliability-based formulation completes this paper.

Deterministic extended H-S principles

We start with basic relations being valid in mechanics of continuum and being appropriate for our next considerations. First, let us distinguish two levels: macroscopic, described by coordinate system *Ox* and microscopic, heterogeneous, described by the system *Oy*. In macroscopic level we have locally homogeneous Hooke's law valid as:

$$\Sigma(x) = \bar{L}(x)E\varepsilon(x) + \bar{\lambda}(x) = \bar{L}(x)(E(x) - \bar{\mu}(x)) \qquad (1)$$

where bar denotes the locally homogeneous quantities, depending on *x*. The last equation can be verbally described as: The overall stress tensor is equal to product of overall stiffness and overall strain tensor plus the homogenized eigenstress. Equivalently it is equal to product of overall stiffness and subtraction of the overall strain and the homogenized eigenstrain. In microscopic level for fixed x similar Hooke's law holds for heterogeneous medium with the stiffness *L*. This means that for microlevel the expression similar to (1) holds as:

$$\sigma(y) = L(y)\varepsilon(y) + \lambda(y) = L(y)(\varepsilon(y) - \mu(y)) \qquad (2)$$

The relation between both levels is expressed as:

$$\int_\Omega \varepsilon d\Omega = Emeas\Omega, \int_\Omega \sigma d\Omega = \Sigma meas\Omega \qquad (3)$$

where *meas* stands for measure. The other denotation is clear from Fig. 1. Introducing stress polarization tensor t defined as:

$$t = (L-D)\varepsilon + \lambda, C(t-\lambda) - \varepsilon = 0 \qquad (4)$$

where D is the stiffness of a comparative medium, *(L-D)C = I*, and I is the unit tensor.

The first primary and the second Hashin-Shtrikman variational principles will first be defined in deterministic form.
The first principle can be formulated as: find the stationary point of the extended functional U defined as:

$$U = Uo - \frac{1}{2}\int_\Omega (C(t-\lambda)(t-\lambda) - 2t\eta - \Delta t - \lambda M\lambda) d\Omega \qquad (5)$$

where

$$\Delta = \varepsilon - \eta$$

and the latter variable is the strain on the comparative medium, being known. U_o is the energy of the comparative medium.

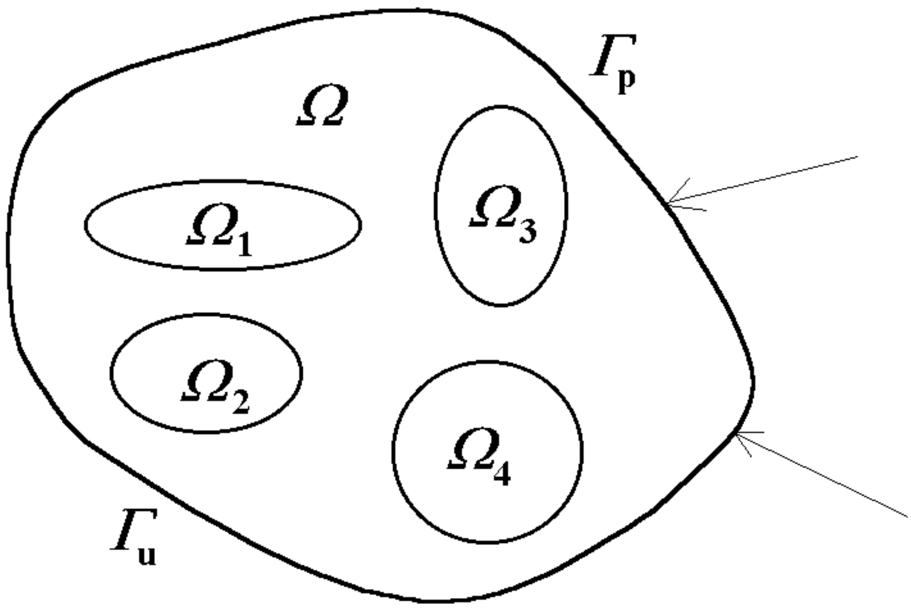

Fig. 1: Geometry and denotation of the trial body

In order to formulate the second extended principle, let us introduce the strain polarization tensor c defined by:

$$c = (M - S)\sigma + M\lambda \tag{6}$$

and S is the compliance of the comparative medium.The second deterministic principle reads as: find the stationary point for the extended functional:

$$U = Uo - \frac{1}{2}\int_{\Omega} (K - L\lambda)(K - L\lambda) - 2cs - s'c)\, d\Omega \tag{7}$$

where s is the known stress on the comparative medium and $s´$ is the difference real stress minus s. U_o is again the energy of the comparison medium. In both cases the stationary values $U´$ and U^o of the functionals are equal to the actual potential energy stored in the heterogeneous and anisotropic body. In the first case it holds for primary and in the second case for the complementary energy density

$$U' = \int_{\Omega} W d\Omega, U^\circ = \int_{\Omega} W^\circ d\Omega \tag{8}$$

where W and Wo are the energy densities. Their relation is depicted in Fig. 2.

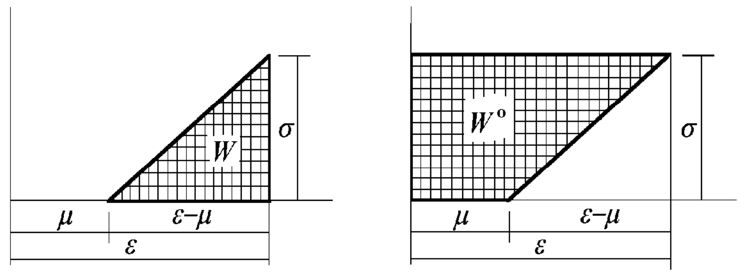

Fig. 2: Potential energy densities and their relations

where

$$\lambda = -L\mu$$

Equivalent formulation to (5) under assumption that the unbounded domain is considered (and the radiation condition holds valid) leads to integral equations (see Willis (1977), and for more precise form see Procházka, Šejnoha (1996)):

$$\varepsilon = \eta - \int_{\Omega} \beta t d\Omega \qquad (9)$$

where the real strain is equal to the strain on the comparative medium minus volume integral over known kernel and stress polarization tensor t . Note that in the last equation convected term is not written for brevity. For more detail see Procházka, Šejnoha (1996). Eliminating real strain from (5) by virtue of (9), we get a simplified principle only for polarization tensor:

$$U(t) = \int_{\Omega} \left((C(t-\lambda)(t-\lambda)) + t \int_{\Omega} \beta t d(\Omega - 2t\eta) \right) d\Omega \qquad (10)$$

Formulation for composites with randomly distributed fibers

Identify by a individual members of a sample space S , define by $p(a)$ the probability density of a in S , and define a characteristic function $k_r(x,a) = 1$ for x in phase r , and = 0 otherwise. Then the probability function $P_r(x)$ of finding phase r at x is:

$$P(x) = \langle k(x) \rangle = \int_{a} k(x;a)p(a)da \qquad (11)$$

and the two point probability $P_{rr'}(x,x')$ of finding simultaneously phase r at x and r' at x' is:

$$P(x, x') = \int_a k(x, a)k(x', a)p(a)da \qquad (12)$$

The polarization field can then be written as:

$$t(x, a) = \sum_{r=1}^{m} t(x)k(x, a) \qquad (13)$$

where m is the number of phases. Further we assume that the material behaves uniformly, i. e. homogeneity and ergodicity is taken into consideration. Thus $P_r(x) = P_r = c_r$, $P_{rr'}(x,x') = P_{rr'}(x - x')$, where c_r is the volume ration of the phase r. The latter relations follow from the assumptions and averaging process over entire body being studied.

The extended H-S variational principle changes as:

$$U = \sum_{r=1}^{m} c \int_\Omega [C(t(x) - \lambda(x))(t(x) - \lambda(x)) - 2\eta t(x)]d(x)$$

$$+\left(\sum_{r=1}^{m}\sum_{r'=1}^{m}\int_\Omega t(x)\left[\int_\Omega \beta(x-x')t(x')P(x-x')dx\right]dx'\right)$$

Averaging the tensor of strain (9), we get:

$$<\varepsilon(x)> = \eta(x) - \sum_{r=1}^{m} c\int_\Omega \beta(x-x')t(x')dx'$$

The condition of stationary taking into account the latter relation can now be improved as:

$$C(t-\lambda)c + \sum_{r=1}^{m}\int_\Omega \beta(x-x')[P(x-x')]cc']t(x')dx' = <\varepsilon>(x)c \quad (14)$$

where $c' = c_{r'}$.

Note that in the last formulas we used the variable x as the entire space (infinite body) has been taken into consideration and from this body local material properties are afterwards derived. This leads to some interesting conclusions for nonlocal constitutive relations.

Conclusions for nonlocal constitutive relations

Averaging (4) with respect to y , we get

$$< \sigma > (x) \; = \; D < \varepsilon > \quad < t > \quad = D < \varepsilon > (x) + \sum_{r=1}^{m} t(x)c \qquad (15)$$

If isotropic distribution of phases (with arbitrary anisotropy and shape), it can be derived, see Willis, Drugan (1996) the formula, the minimum representative volume element can be determined in that the macroscopic response could be sufficiently accurate.

The above introduced formulas are simplified after introducing some assumptions, as are independency of average strain of position. All these assumptions does not change formally the form when introducing eigenparameters.

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