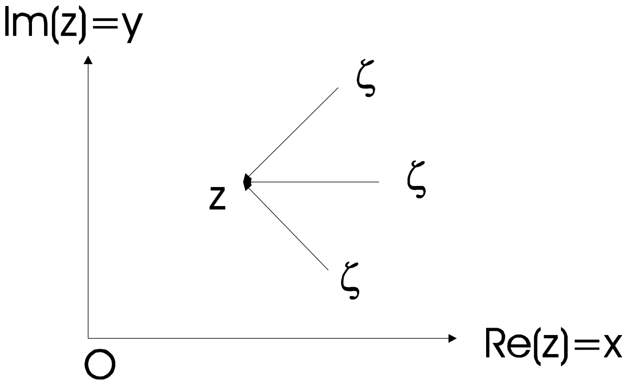
AE 525a/ME 525: Lecture #8

0.1 Analytic functions

Definition 1 *A function $f(z)$ of a complex variable z is said to have a derivative if*

$$\lim_{\zeta \to z} \frac{f(\zeta) - f(z)}{\zeta - z}$$

exists and has the same value regardless of the mode in which $\zeta \to z$. Then this limit is denoted by $f'(z)$ or df/dz.



Different modes of $\zeta \to z$.

Example 2

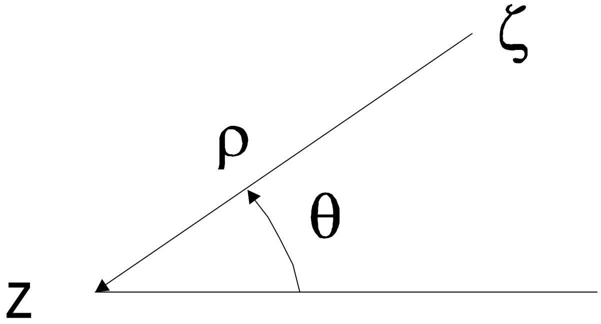
$$\begin{aligned} f(z) &= z^3 \\ f'(z) &= \lim_{\zeta \to z} \frac{\zeta^3 - z^3}{\zeta - z} = \lim_{\zeta \to z} (\zeta^2 + \zeta z + z^2) = 3z^2 \end{aligned}$$

Notation 3 *Formally the definition is the same as in the case of a real variable however, independence of the way in which $\zeta \to z$* ***is very restrictive.***

Example 4

$$\begin{aligned} f(z) &= z\overline{z} \\ \lim_{\zeta \to z} \frac{\zeta\overline{\zeta} - z\overline{z}}{\zeta - z} &= \lim_{\zeta \to z} \frac{\zeta\overline{\zeta} - \overline{\zeta} z + \overline{\zeta} z - z\overline{z}}{\zeta - z} \\ &= \lim_{\zeta \to z} (\overline{\zeta} + z\frac{\overline{\zeta} - \overline{z}}{\zeta - z}) \\ &= \lim_{\zeta \to z} (\overline{\zeta} + ze^{-i2\theta}) \end{aligned}$$

which is different for different θ. Therefore, $f'(z)$ does not exist, except at the origin.



The value of $\zeta - z = \rho e^{i\theta}$ as $\zeta \to z$.

Definition 5 *If $f(z)$ has derivative at a point z_0 and in some neighborhood of z_0, then $f(z)$ is said to be analytic at z_0 (holomorphic, regular).*

Since the definition of the derivative $f'(z)$ is formally the same as for the real variable functions, the rules of differentiation of composite expressions known for real domain variable must hold for complex variable as well, i.e.,

$$\begin{aligned}(af(z)+bg(z))' &= af'(z)+bg'(z)\\ (fg)' &= f'g+fg'\\ (f/g)' &= \frac{f'g-fg'}{g^2}; g\neq 0\\ [f(g(z))]' &= f'(g)g'(z)\end{aligned}$$

Example 6

$$\begin{aligned}\frac{de^z}{dz} &= \lim_{\zeta\to z}\frac{e^\zeta - e^z}{\zeta - z} = \lim_{\zeta\to z} e^z\frac{e^{\zeta - z}-1}{\zeta - z}\\ &= e^z\lim_{\zeta\to z}\frac{1+(\zeta - z)+\frac{(\zeta - z)^2}{2!}+...}{\zeta - z}\\ &= e^z\lim_{\zeta\to z}[1+\frac{\zeta - z}{2!}+...] = e^z\end{aligned}$$

Example 7 *Similarly, we can show that*

$$\begin{aligned}\frac{d}{dz}\sin z &= \lim_{\zeta\to z}\frac{\sin(\zeta - z+z)-\sin(z)}{\zeta - z} = \cos z\\ \frac{d}{dz}\ln z &= \frac{1}{z}\\ \frac{d}{dz}\tan^{-1} z &= \frac{1}{1+z^2}, \quad \textit{etc.}\end{aligned}$$

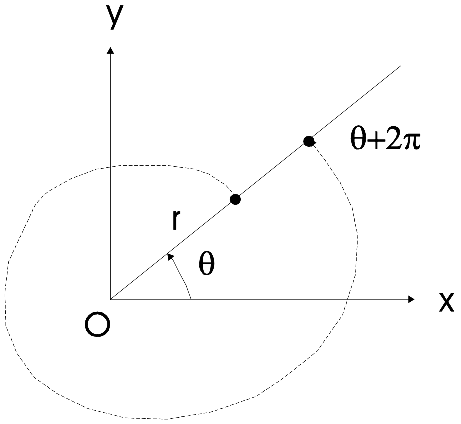
0.2 Branch Cuts and Riemann Surfaces

Definition 8 *Riemann surface is a generalization of the* $z-$*plane to a surface of more then one sheet such that a multiple-valued function has only one value corresponding to each point on that sheet.*

Example 9 *Consider the function*

$$\begin{aligned} f(z) &= z^{1/2} = w \\ z &= re^{i(\theta+2n\pi)}; n = 0,1,2,... \Rightarrow \\ w &= \sqrt{r}e^{i(\theta/2+n\pi)} \Rightarrow \\ n &= 0 \Rightarrow w = \sqrt{r}e^{i\theta/2} = w_1 \\ n &= 1 \Rightarrow w = -\sqrt{r}e^{i\theta/2} = w_2 \\ n &= 3 \Rightarrow w = \sqrt{r}e^{i\theta/2} = w_1, \quad \textit{etc.} \end{aligned}$$

Therefore, there will be two possible values of the function w_1 *and* w_2, *called the* ***branches of*** w *as* θ *increases in multiples of* 2π.

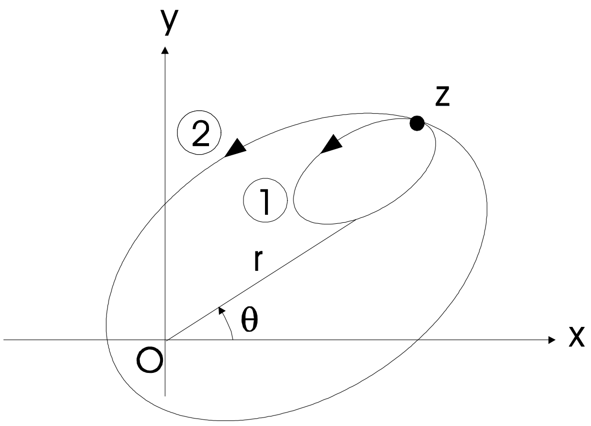


Increase of the argument θ in increments of 2π.

Therefore, for the function $f(z) = z^{1/2}$ there are two types of paths:

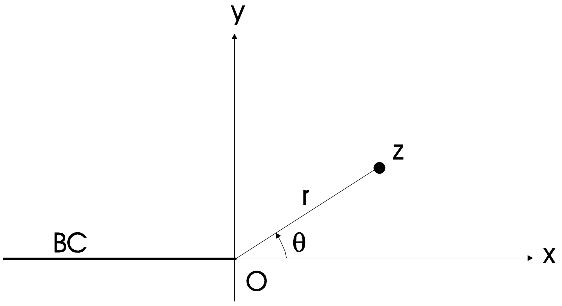
- Path 1, where θ never increases for 2π, and
- path 2, where θ increases beyond 2π resulting in double values w_1 and w_2.

Thus the origin possesses the property that a loop about it **interchanges the branches** and thus the origin O is defined as **branch point** of the function.



Different paths for the function $z^{1/2}$.

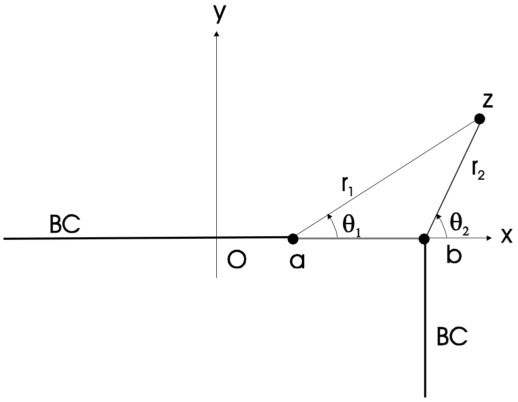
To prevent the multiple valuedness we introduce a **branch cut.** For example, for $w = z^{1/2}$ the branch cut is depicted by the following figure



A branch cut for the function $z^{1/2}$.

If we choose the value of w at z_0 to be, say, $w_1(z_0)$ and require that the value of w at any other point z_1 be obtained by **continuous variation of** w along any curve joining z_0 and z_1, then the fact that loops about the origin are prevented means that $z^{1/2}$ is uniquely defined.

Example 10 *The function $f(z) = \sqrt{(z-a)(z-b)}$, where a and b are real has branch points at $z = a$ and $z = b$. Let's define the branch cuts as*



Branch cuts for $\sqrt{(z-a)(z-b)}$.

which implies that

$$\begin{aligned} z-a &= r_1 e^{i\theta_1}; -\pi < \theta \leq \pi \\ z-b &= r_2 e^{i\theta_2}; -\frac{\pi}{2} \leq \theta < \frac{3\pi}{2} \end{aligned}$$

Then the function

$$f(z) = \sqrt{r_1 r_2} e^{i(\theta_1/2+\theta_2/2)}$$

is uniquely defined in the complex plane.

Example 11 *Consider the function* $f(z) = (z-a)^{1/n_1}(z-b)^{1/n_2}$ *with branch points at* $z = a$ *and* $z = b$. *Using*

$$\begin{aligned} z-a &= r_1 e^{i(\theta_1+2n\pi)} \\ z-b &= r_2 e^{i(\theta_2+2n\pi)} \end{aligned}$$

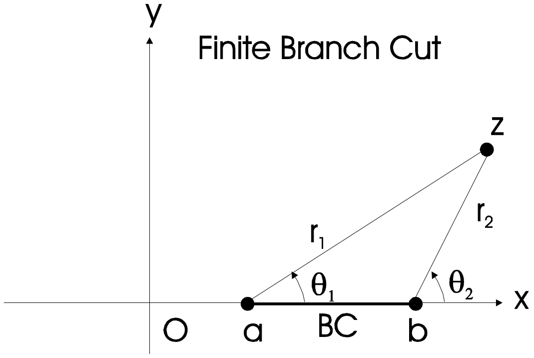
we have that

$$f(z) = r_1^{1/n_1} r_2^{1/n_2} e^{i[\theta_1/n_1+\theta_2/n_2+2n\pi(1/n_1+1/n_2)]}$$

If

$$\frac{1}{n_1} + \frac{1}{n_2} = \pm m$$

where is an integer, then the finite branch cut is possible and the function is single valued in the complex plane as shown.



Finite branch cut for the function $(z-a)^{1/n_1}(z-b)^{1/n_2}$ *when* $1/n_1 + 1/n_2 = \pm m$*(an integer).*

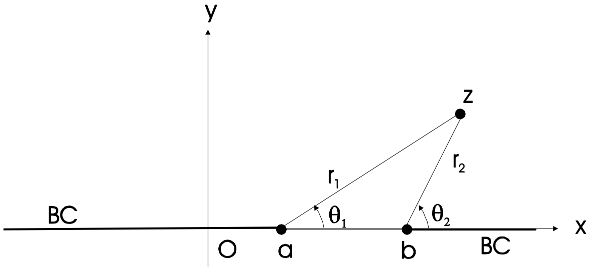
Example 12 *Consider the function*

$$\begin{aligned} f(z) &= \ln[(z-a)(z-b)] \\ &= \ln(r_1 r_2) + i(\theta_1 + \theta_2 + 4n\pi) \end{aligned}$$

where

$$\begin{aligned} z-a &= r_1 e^{i(\theta_1+2n\pi)} \\ z-b &= r_2 e^{i(\theta_2+2n\pi)} \end{aligned}$$

which has the branch points at $z=a$ *and* $z=b$*. Obviously, the finite branch cut is unacceptable since we are not allowed to increase* θ_1 *or* θ_2 *for* 2π *which would introduce multiple-valuedness of* f*. Thus the following branch cuts are acceptable*



Branch cuts for the function $\ln[(z-a)(z-b)]$.

Example 13 *Consider the function*

$$f(z) = \ln\frac{z-a}{z-b} = \ln\frac{r_1}{r_2} + i(\theta_1 - \theta_2)$$

where

$$\begin{aligned} z-a &= r_1 e^{i(\theta_1+2n\pi)} \\ z-b &= r_2 e^{i(\theta_2+2n\pi)} \end{aligned}$$

has the branch points at $z = a$ and $z = b$. Therefore, the finite branch cut is OK.

Example 14 *For the function*

$$f(z) = z^{\frac{p}{q}} = (z^{\frac{1}{q}})^p$$

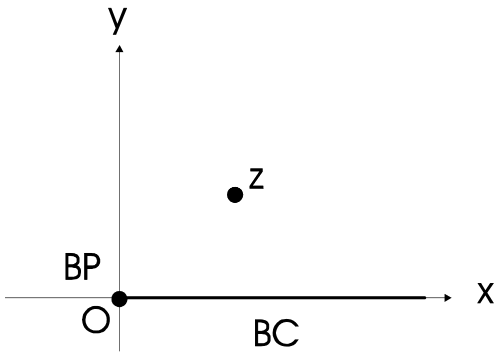
has a branch point at O and a semiinfinite branch cut ai acceptable. For example,

$$f(z) = z^{\frac{1}{n}}$$

where n is an integer can be represented as

$$f(z) = r^{\frac{1}{n}} e^{i(\theta/n + 2m\pi/n)}$$

and the branch cut can be chosen as



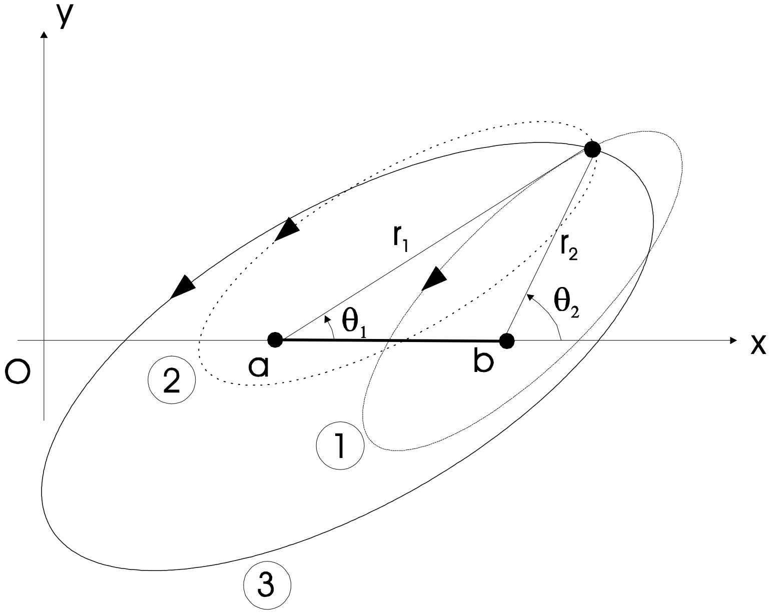
Branch cut for $f(z) = z^{1/n}$.

Notation 15 *We should note that both the function and all its derivatives $f(z) = z^{1/n}$ have algebraic branch point at O. However, the function $f(z) = \ln z$ has a logarithmic branch point at O while its derivatives do not have logarithmic branch point.*

Example 16 *Let's reconsider the function*

$$f(z) = \sqrt{(z-a)(z-b)} = r_1^{1/2} r_2^{1/2} e^{i[\theta_1/2 + \theta_2/2 + n\pi + m\pi]}$$

as the complex variable z varies along various closed loops in the $z-$plane:



Different paths for the function $\sqrt{(z-a)(z-b)}$.

Thus we have

$$\begin{aligned} path \quad 1 &: \quad \theta_2 \to \theta_2 + 2\pi; \theta_1 \to \theta_1 \Rightarrow f(z) \to -f(z) \\ path \quad 2 &: \quad \theta_2 \to \theta_2; \theta_1 \to \theta_1 + 2\pi \Rightarrow f(z) \to -f(z) \\ path \quad 3 &: \quad \theta_2 \to \theta_2 + 2\pi; \theta_1 \to \theta_1 + 2\pi \Rightarrow f(z) \to f(z) \end{aligned}$$

Therefore, the finite branch cut is OK.

0.3 Cauchy-Riemann Equations

Consider an analytic function $f(z) = u(x,y) + i(v(x,y)$. Then,

$$\begin{aligned} f'(z_0) &= \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} \\ &= \lim_{z\to z_0} \frac{[u(x,y) - u(x_0,y_0)] - i[v(x,y) - v(x_0,y_0)]}{(x-x_0) + i(y-y_0)} \end{aligned}$$

Since the mode of z approaching z_0 arbitrary, we choose first that along the line y_0, i.e., from point (x_0, y_0) to (x, y_0). Therefore,

$$\begin{aligned} f'(z_0) &= \lim_{z\to z_0} \left\{ \frac{u(x,y) - u(x_0,y_0)}{(x-x_0)} + i\frac{v(x,y) - v(x_0,y_0)}{(x-x_0)} \right\} \\ &= \frac{\partial u(x_0,y_0)}{\partial x} + i\frac{\partial v(x_0,y_0)}{\partial x} \end{aligned}$$

Similarly, by taking the path from (x_0, y_0) to (x_0, y) we have that

$$\begin{aligned} f'(z_0) &= \lim_{z\to z_0}\left\{\frac{u(x_0,y)-u(x_0,y_0)}{i(y-y_0)}+i\frac{v(x_0,y)-v(x_0,y_0)}{i(y-y_0)}\right\} \\ &= -i\frac{\partial u(x_0,y_0)}{\partial y}+\frac{\partial v(x_0,y_0)}{\partial y} \end{aligned}$$

Therefore, we get the Cauchy-Riemann equations

$$\begin{aligned} \frac{\partial u(x,y)}{\partial x} &= \frac{\partial v(x,y)}{\partial y} \\ \frac{\partial u(x,y)}{\partial y} &= -\frac{\partial v(x,y)}{\partial x} \end{aligned}$$

which means that if the derivative $f'(z)$ exists, then the Cauchy-Riemann equations are satisfied. Thus the Cauchy-Riemann equations are necessary condition for existence of derivative $f'(z)$.

Proposition 17 *The Cauchy-Riemann equations are sufficient condition for existence of a derivative.*

Proof. Sufficient condition for existence of derivative $f'(z)$ (i.e., if $u_x = v_y \& u_y = -v_x \Rightarrow f'(z)$ exists). From Taylor series we have that

$$\begin{aligned} u(x,y)-u(x_0,y_0) &= u_x(x_0,y_0)(x-x_0)+u_y(x_0,y_0)(y-y_0)+... \\ v(x,y)-v(x_0,y_0) &= v_x(x_0,y_0)(x-x_0)+v_y(x_0,y_0)(y-y_0)+... \end{aligned}$$

and consequently

$$\begin{aligned} &\lim_{z\to z_0}\left\{\frac{u(x,y)-u(x_0,y_0)+i[v(x,y)-v(x_0,y_0)]}{(x-x_0)+i(y-y_0)}\right\} \\ =\ &\lim_{z\to z_0}\left\{\frac{u_x(x_0,y_0)(x-x_0)+u_y(x_0,y_0)(y-y_0)+...}{(x-x_0)+i(y-y_0)}\right\} \\ &+i\lim_{z\to z_0}\left\{\frac{v_x(x_0,y_0)(x-x_0)+v_y(x_0,y_0)(y-y_0)+...}{(x-x_0)+i(y-y_0)}\right\} \\ =\ &\lim_{z\to z_0}\left\{\frac{u_x(x_0,y_0)(x-x_0)-v_x(x_0,y_0)(y-y_0)+...}{(x-x_0)+i(y-y_0)}\right\} \\ &+\lim_{z\to z_0} i\left\{\frac{v_x(x_0,y_0)(x-x_0)+u_x(x_0,y_0)(y-y_0)+...}{(x-x_0)+i(y-y_0)}\right\} \\ =\ &u_x(x_0,y_0)+iv_x(x_0,y_0)=f'(z_0) \end{aligned}$$

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Thus we have proved that Cauchy-Riemann equations are necessary and sufficient condition for analyticity.

Notation 18 *We should note that from the Cauchy-Riemann equations it follows that*

$$\begin{aligned} u_x &= v_y \& u_y = -v_x \Rightarrow \\ u_{xx} + u_{yy} &= 0 = \nabla^2 u \\ v_{xx} + v_{yy} &= 0 = \nabla^2 v \end{aligned}$$

or that real and imaginary parts of an analytical function are harmonic.

Example 19 *No purely real function can be analytic unless it is a constant. Namely, let* $\mathrm{Im}(f(z) = v(x,y) = 0$ *or*

$$f(z) = u(x,y)$$

If $f(z)$ is analytic it must satisfy the Cauchy-Riemann equations, thus

$$\begin{aligned} u_x &= v_y = 0 \Rightarrow u(x) = g(y) \\ u_y &= -v_x = 0 \Rightarrow g'(y) = 0 \Rightarrow g(y) = const \end{aligned}$$

so

$$u(x) = g(y) = const$$

0.4 Integration in the Complex Plane

Consider a path C in a plane of a complex variable z. Let $f(z)$ be any complex function (analytic or not). Then

$$\int_C f(z)dz \equiv \lim_{n\to\infty;|\Delta z_j|\to 0} \sum_{j=1}^{n} f(\zeta_j)\Delta z_j \tag{1}$$

this limit is called integral of $f(z)$ along C, provided that the limit does not depend on the way in which z_j and ζ_j are chosen. In terms of x and y we have that

$$\int_C f(z)dz = \int_C udx - vdy + i\int_C fdx + udy$$

so that for these integrals the real variable theory applies.

Immediate consequence of definition (1) is that the usual rules for manipulation of integrals apply:

$$\int_{C:AB} f(z)dz = -\int_{C:BA} f(z)dz$$

where $C : AB$ denotes moving along the curve C from points A to B. Similarly

$$\int_C [af(z) + bg(z)]dz = a\int_C f(z)dz + b\int_C g(z)dz, \quad etc.$$

Notation 20 *If $f(z)$ is bounded on C, i.e., $\mid f(z) \mid\leq M, z \in C$, then the triangle inequality can be applied to (1) to obtain the following result*

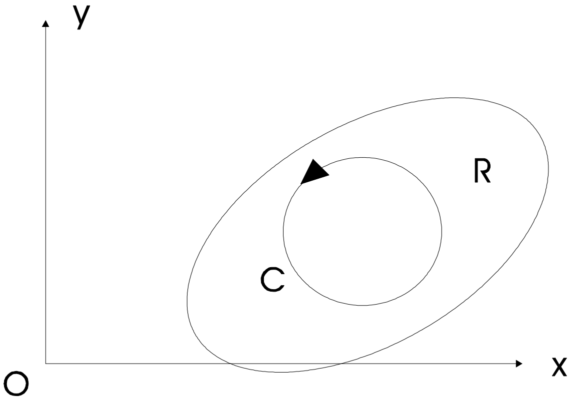
$$\mid \int_C f(z)dz \mid\leq \int_C |f(z)||dz| \leq ML$$

where L denotes the length along the curve of integration C.

Theorem 21 *Cauchy Theorem. Let $f(z)$ be an analytic function in R, and let C denotes a closed curve completely contained in R. Then*

$$\oint_C f(z)dz = 0 \tag{2}$$

where unless stated differently the integration along C takes place in counter-clockwise direction.



Integration path in Cauchy theorem.

Proof. Recall first the application of divergence theorem to a region A bounded by the closed curve C

$$\int_A div\mathbf{u}dA = \oint_C \mathbf{u}\cdot\mathbf{n}ds$$

for a vector-valued function $\mathbf{u} = (u, v)$ results in

$$\int_A (u_x + v_y)dA = \oint_C (un_x + vn_y)ds$$

where $\mathbf{n}$ denotes an outward unit normal to C and s denotes the length of the curve. It is easy to show that

$$\begin{aligned} dsn_y &= -dx \\ dsn_x &= dy \end{aligned}$$

so that

$$\int_A (u_x+v_y)dA = \oint_C (un_x+vn_y)ds = \int_A (u_x+v_y)dA = \oint_C (udy-vdx) \equiv Green's \quad theorem$$

Therefore

$$\int_A (u_x - v_y)dA = \oint_C (udy + vdx)$$

Consequently

$$\begin{aligned} \oint_C f(z)dz &= \oint_C (u + i * v)(dx + i * dy) \\ &= \oint_C (udx - vdy) + i \oint_C (vdx + udy) \\ &= -\oint_C (v_x + u_y)dA + i \oint_C (u_x - v_y)dA = 0 \end{aligned}$$

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Notation 22 *The proof Green's theorem requires continuity of partial derivatives of* u *and* v *(i.e.,* $f'(z)$ *continuous). Goursat proved Cauchy theorem under less restricted conditions.*

0.5 Multiple Connected Regions

Definition 23 *A simply connected region is defined as the region in which a closed curve drawn in the region encloses only the points belonging to the region.*

For a region R not simply connected and for $f(z)$ analytic and single valued in R we have that

$$\oint_{C_1} f(z)dz \neq 0$$

Namely, by introducing appropriate cuts, the region becomes simply connected. Therefore we get that the so called **generalization of Cauchy Theorem**

$$\oint_{C_1} f(z)dz - \oint_{C_2} f(z)dz - \oint_{C_3} f(z)dz = 0$$

where the integration along closed loops takes place in the counterclockwise direction.

Example 24 *Show that*

$$\oint_C \frac{dz}{z} = 2\pi i$$

for any closed curve C *enclosing the origin.*

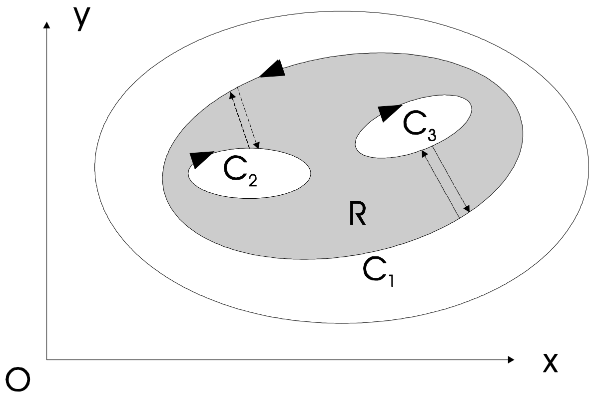
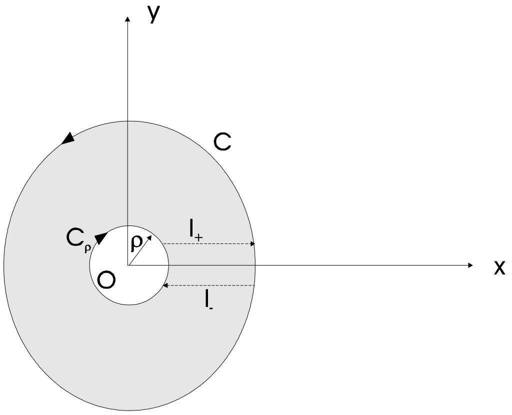


Figure 1: Multiple connected region R.



Integration contour for $\oint_C dz/z$.

First we isolate the singularity by excluding a small circle C_ρ *of radius* ρ *from the domain. Therefore, with appropriate cuts the new domain is simply connected and the function* $1/z$ *is single valued. Thus the generalize Cauchy Theorem results in*

$$\oint_C \frac{dz}{z} = \lim_{\rho\to 0} \oint_{C_\rho} \frac{dz}{z} = \lim_{\rho\to 0} \int_0^{2\pi} \frac{i\rho e^{i\theta}}{\rho e^{i\theta}} d\theta = 2\pi i$$

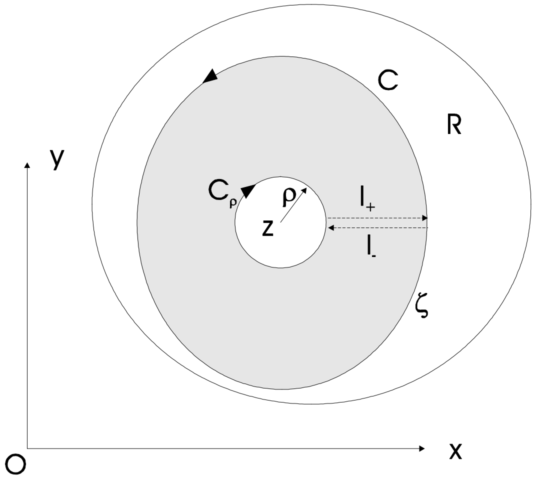
0.6 Cauchy's Integral Formula

Cauchy's integral formula expresses the value of a function $f(z)$ at any point inside a contour in terms of its values on the contour.

Theorem 25 *Cauchy's Integral Formula. If* $f(z)$ *is analytic in a simply con-*

nected region R and if C is any closed contour in R, then

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} d\zeta$$



Proof. Let's isolate the singularity of the integrand at $\zeta = z$ by removing an infinitesimal circle C_ρ centered at z.Then the integrand becomes analytic and single valued everywhere inside the contour $C + C_\rho + l_+ + l_-$. Consequently, Cauchy theorem implies that

$$\begin{aligned}
\oint_C \frac{f(\zeta)}{\zeta - z} d\zeta &= \lim_{\rho \to 0} \oint_{C_\rho} \frac{f(\zeta)}{\zeta - z} d\zeta \\
&= \lim_{\rho \to 0} \oint_{C_\rho} \frac{f(z) + f(\zeta) - f(z)}{\zeta - z} d\zeta \\
&= \lim_{\rho \to 0} \oint_{C_\rho} \frac{f(z)}{\zeta - z} d\zeta + \lim_{\rho \to 0} \oint_{C_\rho} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta
\end{aligned}$$

But using the result from the previous example we have that

$$\lim_{\rho \to 0} \oint_{C_\rho} \frac{f(z)}{\zeta - z} = f(z) 2\pi i$$

We shall prove later that if $f(z)$ has derivative at $z = z_0$ then it is also continuous at z_0, i.e., $\mid f(z) - f(z_0) \mid < \varepsilon$ for $z \to z_0$. Thus as $\rho \to 0$

$$\oint_{C_\rho} \mid \frac{f(\zeta) - f(z)}{\zeta - z} \mid\mid d\zeta \mid \leq \int_0^{2\pi} \frac{\varepsilon}{\rho} \rho d\theta = 2\pi\varepsilon \to 0$$

as $\rho \to 0$. Therefore,

$$\oint_C \frac{f(\zeta)}{\zeta - z} d\zeta = 2\pi i f(z)$$

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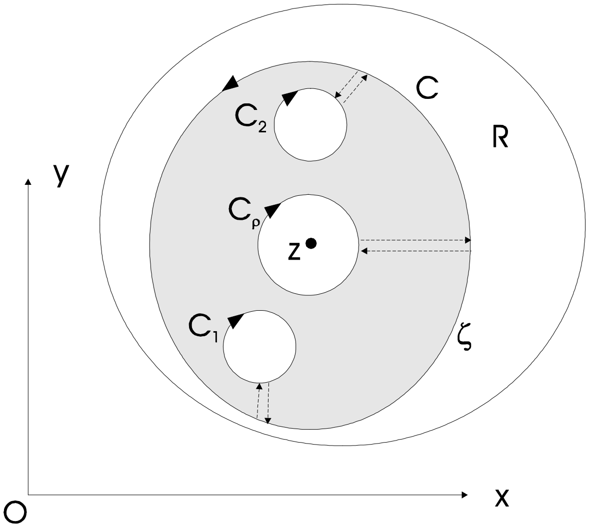
Notation 26 *If z is outside the contour C,then $F(\zeta)/(\zeta - z)$ is analytic in R and thus*

$$\oint_C \frac{f(\zeta)}{\zeta - z} d\zeta = 0$$

0.6.1 Case of Multiple Connected Regions

Lets consider the multiple connected domain as shown bellow.



Then with appropriate cuts, the domain becomes simply connected and the Cauchy integral formula becomes

$$\begin{aligned} & \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i}\oint_{C_2} \frac{f(\zeta)}{\zeta - z} d\zeta \\ = & \frac{1}{2\pi i} \lim_{\rho \to 0} \oint_{C_\rho} \frac{f(\zeta)}{\zeta - z} d\zeta = f(z) \end{aligned}$$

So that the corresponding Cauchy integral formula becomes

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i}\oint_{C_2} \frac{f(\zeta)}{\zeta - z} d\zeta$$

0.7 Integrals Dependent on a Parameter

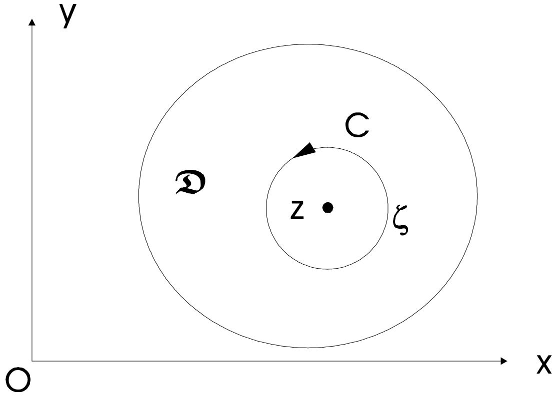
Recall the Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i}\oint_c \frac{f(\zeta)}{\zeta - z} d\zeta$$

where ζ is a variable of integration and z is a parameter. In general, we consider

$$\begin{aligned} F(z) &= \oint_C \phi(z,\zeta)d\zeta \qquad (3) \\ z &= x+iy; \quad \zeta = \xi + i\eta \end{aligned}$$

and C is a closed piecewise smooth curve.



The function ϕ is the function of two complex variables satisfying the following properties:

1. $\phi(z,\zeta)$ is analytic for $z \in D$ and $\zeta \in C$.
2. ϕ and $\partial\phi/\partial z$ are continuous functions of z, ζ for $z \in D$ and $\zeta \in C$.

Claim 27 *Then the function $F(z)$ is an analytic function of the complex variable z in D and derivative of $F(z)$ may be computed by differentiating under the integral sign.*

Proof. Consider

$$\begin{aligned} F(z) &= \oint_C \phi(z,\zeta)d\zeta = \oint_C (u+iv)(d\xi + id\eta) \Rightarrow \\ F(z) &= U(x,y) + iV(x,y) = \oint_C (ud\xi - v\eta) + i\oint_C (vd\xi + ud\eta) \\ &\therefore \;\; U(x,y) = \oint_C (ud\xi - v\eta); V(x,y) = \oint_C (vd\xi + ud\eta) \end{aligned}$$

Now the functions u and v possess partial derivatives w.r. to x and y and are continuous in x and y. Thus, the partial derivatives w.r. to x and y of U exists and may be computed by differentiating under the integral sign. Then

$$\begin{aligned} U_x &= \oint_C (u_x d\xi - v_x \eta) \\ U_y &= \oint_C (u_y d\xi - v_y \eta) \end{aligned}$$

are continuous functions of x, y in D.Similarly,

$$\begin{aligned} V_y &= \oint_C (v_y d\xi + u_y d\eta) = \oint_C (u_x d\xi - v_x d\eta) = U_x \\ V_x &= \oint_C (v_x d\xi + u_x d\eta) = \oint_C (-u_y d\xi + v_y d\eta) = -U_y \end{aligned}$$

or the functions U and V satisfy the Cauchy-Riemann equations. Consequently, the function $F(z) = U + iV$ is analytic in D. Note that

$$\begin{aligned} F'(z) &= U_x + iV_x = \oint_C (u_x d\xi - v_x \eta) + i \oint_C (v_x d\xi + u_x d\eta) \\ &= \oint_C (u_x + iv_x)(d\xi + i\eta) = \oint_C \frac{\partial \phi}{\partial z} d\zeta \end{aligned}$$

Thus to compute the derivative of an integral $F(z) = \oint_C \phi(z, \zeta) d\zeta$, (when ϕ is analytic for $\zeta \in C$ and $z \in D$, and ϕ and $\partial\phi/\partial z$ are continuous functions of ζ and z) one differentiates the integrand w.r to parameter z. ■