ALTERNATIVE METHODS FOR CALCULATING THE FUZZY LIMIT MATRIX IN THE ANALYTIC NETWORK PROCESS

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Abstract:

In this paper we suggest a Markov based approach for calculating the fuzzy limit matrix in the analytic network process, where we experimented with several alternative computational methods that accommodate fuzziness in the supermatrix calculations and allow for considering cumulative uncertainty. The methods include constrained fuzzy computations (closed form and numerical methods), interval computations and heuristic based methods. A numerical example is given to demonstrate the ability of the methods and compare the results with supermatrix computations with crisp values.

Keywords: Analytic Network Process, Supermatrix, Fuzzy Limit Matrix, Fuzzy Arithmetic, Interval Arithmetic, Fuzzy Inverse

1. Introduction

The analytic network process (ANP) is a multi-criteria theory of measurement used to derive priority scales from individual or group judgments representing the relative dependence among the elements of a network system. To capture the transmission of influence along all paths defined in the network and to obtain the overall priorities of the elements, a supermatrix is formed and analyzed (Saaty, 2001). This partitioned matrix contains local priorities derived from pairwise comparisons throughout the network and thus serves as a unifying framework. The cumulative influence of each element on every other element with which it interacts, i.e. its global priority, is determined by raising the supermatrix to limiting powers.

A decision making method should provide the flexibility to deal with problems in which descriptions of activities, observations and judgments are by nature subjective, vague and imprecise. There is an extensive literature on decision making that addresses the situation where the comparison ratios are imprecise judgments (Leung and Cao, 2000). In decision problems, it may become unrealistic and infeasible to acquire exact judgments in pairwise comparisons, given the complexity and uncertainty of the problem, along with the inherent subjective nature of human judgments. Such source of imprecision in real complex decision making is inevitable and this type of uncertainties can be better handled using fuzzy set theoretic approaches. Decision-makers usually find it more convenient to express interval judgments than fixed value judgments (Bozdag et al., 2003). Here, fuzzy-based techniques can be viewed as a generalized form of interval analysis that addresses uncertain and/or imprecise information. Furthermore, decision problems may involve not only quantitative factors but also qualitative factors that are often accompanied by ambiguity and vagueness. The involvement of fuzzy theory can adequately resolve the ambiguity and vagueness and derive meaningful priorities explicitly from the complex decision structure (Asan and Soyer, 2010).

There are several fuzzy ANP methods suggested in the literature. The way they cope with uncertain judgments is by expressing the comparison ratios as fuzzy numbers, which incorporate the imprecise and subjective nature of human judgments (Chen et al. 1992; Wu et al., 2008; Razmi et al. 2009; among others). However, these fuzzy methods have a common drawback that prevents them to observe the influence of uncertainty on the global (limiting) priorities. They perform supermatrix computations with nonfuzzy numbers, though a fuzzy set theoretic approach to the supermatrix would provide the
opportunity to capture the uncertainty associated with the cumulative influence of each element on every other element with which it interacts in the network. The reason for the preference of crisp values in the supermatrix computations is the problem of convergence resulting from standard fuzzy arithmetic operations (Buckley and Eslami, 2002; Huang, 2008). The power sequence of a fuzzy matrix (i.e. fuzzy matrix multiplication) produces global priority values in form of fuzzy numbers which are unsatisfactory. Let a fuzzy matrix be given as:

$$\mathbf{M} = \begin{bmatrix}
(0.3, 0.4, 0.5) & (0.7, 0.8, 0.9) \\
(0.5, 0.6, 0.7) & (0.1, 0.2, 0.3)
\end{bmatrix}$$

Then, the limiting matrix of $\mathbf{M}$ can be calculated using standard fuzzy arithmetic as:

$$\lim_{k \to \infty} \mathbf{M}^{(k)} = \begin{bmatrix}
(0, 0.5714, \infty) & (0, 0.5714, \infty) \\
(0, 0.4285, \infty) & (0, 0.4285, \infty)
\end{bmatrix},$$

where $(k)$ denotes the power operation. This unsatisfactory result is caused by the fact that the end points (lower and upper bounds) of the fuzzy numbers in the supermatrix do not follow the properties of a stochastic matrix - i.e. the sum of all columns in terms of the end points do not equal to one as the midvalues.

In this paper we suggest a Markov based approach, where we experimented with several alternative computational methods that accommodate fuzziness in the supermatrix calculations and allow for considering cumulative uncertainty. The methods include constrained fuzzy computations (closed form and numerical methods), interval computations and heuristic based methods. A numerical example is given to demonstrate the ability of the methods and compare the results with supermatrix computations with crisp values.

2. The Fuzzy ANP

Fuzzy set theory has been developed to meet the objective of solving problems in which descriptions of activities, observations and judgments are by nature subjective, vague and imprecise. In general, the term ‘fuzzy’ refers to the situation in which no boundary for the set of observations or judgments can be well defined (Chen et al., 1992). The theory provides numerous methods to represent the qualitative judgment of the decision maker as quantitative data. In this study, due to their simplicity in both design and implementation based on little information, triangular fuzzy numbers are used to assess the preferences of decision makers. Note that, for TFNs, addition and subtraction are closed operations, however multiplication and division of TFNs only produce approximate TFNs. The steps of the proposed Fuzzy ANP method, based on some suggestions in Liou and Wang (1992), Buckley and Eslami (2002), Wang et al. (2008), Huang (2008) and Kirytopoulos et al. (2011) for calculating the local and global priority vectors, are as follows.

2.1. Local Priorities

First, judgments are elicited and aggregated to establish the fuzzy pairwise comparison matrices. Decision makers are asked to perform a series of pairwise comparisons where $(a_{nm}^{ij})$ denotes the relative dominance of the $n_i$th element compared to the $n_k$th element in the $i$th node with respect to a common aspect/criterion $m_j$ in the $j$th node. Since it can successfully deal with inhomogeneous evaluations and runaway values, we suggest using the geometric mean algorithm to aggregate the fuzzy comparison judgments of different decision makers. Once the aggregate fuzzy pairwise comparison matrices representing the dominance relations given in the decision problem are established, the local priority
vectors are estimated using a fuzzy version of the additive normalization method. The value of an element’s fuzzy local priority with respect to a certain aspect \( m_j \), defined as 
\[
\tilde{s}_{ni}^j = \frac{\sum_{n=1}^{u_{ni}} j_{mj} \sum_{n=1}^{m_{nj}} l_{ni}}{\sum_{n=1}^{u_{ni}} j_{mj} + \sum_{n=1}^{m_{nj}} u_{ni}}
\]
(1)

Fuzzy priority vectors are derived for all fuzzy comparison matrices. Note that, the accuracy of the approximation is increased when a pairwise comparison matrix has a low consistency ratio.

2.2. Global Priorities

A decision problem structured in the form of a network certainly involves complex interactions among the decision elements. In other words, the interactions are not restricted to direct impacts; there may be many indirect impacts between the elements. The supermatrix allows us to analyze and synthesize these indirect effects and complex interactions existing among the elements by a logical procedure where the scales, i.e. local priorities, are added and multiplied by raising the supermatrix to powers (Saaty, 2004; Asan & Soyer, 2009). For example, all the second order impact totals can be obtained from the square of the supermatrix. Thus, we need to compute the limiting power of a supermatrix. However to avoid the convergence problem caused by raising a fuzzy supermatrix to powers, a Markov-based approach is suggested (Buckley and Eslami, 2002; Huang, 2008). The limiting power of a crisp supermatrix reaches an equilibrium distribution, as in the Markov chains. Elements in the network model can be rank ordered by using limiting priorities obtained from the asymptotic steady state distribution of the supermatrix (Büyükyazıcı and Sucu, 2003). The stochastic fuzzy supermatrix \( \tilde{S} \), populated by the normalized fuzzy local priority vectors (fuzzy transition probabilities), is defined as follows

\[
\tilde{S} = \begin{bmatrix}
S_{11} & \cdots & S_{1q} \\
\vdots & \ddots & \vdots \\
S_{q1} & \cdots & S_{qq}
\end{bmatrix}
\]

The existence and uniqueness of the limit matrix is guaranteed if the supermatrix is irreducible and acyclic. For further detail on the necessary properties of the limit supermatrix the reader should refer to Saaty (2001) and Saaty (2004). If the transition matrix is irreducible and acyclic, then \( \tilde{S} \), raised to an arbitrary large power, should tend to an ergodic matrix \( \tilde{E} \). This matrix has similar columns, i.e.,

\[
\lim_{k \to \infty} \tilde{S}^k = \tilde{E} = \tilde{s}_{ni} \cdot \tilde{1}
\]

where \( \tilde{1} \) is a q-dimensional fuzzy raw vector containing ones, and \( \tilde{s}_{ni} \) is a fuzzy column vector whose entries are positive and midvalues sum to 1. Here, \( \tilde{s}_{ni} \) will be regarded as the fuzzy global priority vector (Huang, 2008) satisfying the following equation

\[
\tilde{s} \cdot \tilde{s}_{ni} = \tilde{s}_{ni} \quad \text{or} \quad (I - \tilde{S}) \cdot \tilde{s}_{ni} = \tilde{0}
\]

where \( \tilde{0} \) is a column vector containing zeros. Obviously solving Eq. (4) will not yield a unique solution. The following additional constraint, which describes that the sum of the steady state probabilities equals
one, is necessary to guarantee the uniqueness of the solution $\tilde{s}_{i,n_i}' \cdot \mathbf{1} = \mathbf{1}$. In order to take into account the additional constraint, the last row of the matrix $\bar{I} - \bar{S}$ is replaced by a row of ones and the right hand side is replaced by a column vector containing zeros except its last element which is one (cf. Kirytopoulos et al. 2011). Then, Eq. (4) can be represented as the following fully fuzzy linear system in matrix form

$$\tilde{A} \cdot \tilde{s}_{i,n_i} = \tilde{\mathbf{e}}$$  \hspace{1cm} (5)

where

$$\tilde{A} = \begin{bmatrix}
1 - \tilde{s}_{11}' & \ldots & \ldots & \ldots & -\tilde{s}_{1q}' \\
-\tilde{s}_{11}' & 1 - \tilde{s}_{12}' & \ldots & \ldots & \ldots & -\tilde{s}_{1q}' \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
1 & 1 & \ldots & 1 & \ldots & \ldots & 1
\end{bmatrix}, \quad \tilde{s}_{i,n_i} = \begin{bmatrix}
\tilde{s}_{1,n_1} \\
\tilde{s}_{2,n_2} \\
\vdots \\
\tilde{s}_{q,n_q}
\end{bmatrix}, \quad \text{and} \quad \tilde{\mathbf{e}} = \begin{bmatrix}
(0, 0, 0) \\
(0, 0, 0) \\
\vdots \\
(1, 1, 1)
\end{bmatrix}$$  \hspace{1cm} (6)

It should be noted that the resulting fully fuzzy linear system (FFLS) can be solved by exploiting different computational methods, supposing that $\tilde{A}$ is a nonsingular matrix.

2.2.1. Alternative Methods for Calculating the Fuzzy Global Priority Vector

A computational method for solving a fully fuzzy linear system using Zadeh’s extension principle is definitely not in hand (Dehghan et al., 2006). It is proved that finding all of the real solutions which satisfy a system with interval coefficients is NP-hard. The same result can similarly be derived for fuzzy systems (Dehghan et al., 2006). Recently, many studies have investigated the solution of FFLSs, however very few methods are available for the practical solution. Especially methods which are able to solve systems with matrices including both negative and positive coefficients - as is the case in this study- and guarantee non-negative solutions are a few. Our purpose is to discuss some of these methods which promise a satisfactory solution to the arbitrary FFLS problem. The methods can be classified into the following categories: constrained fuzzy computations (closed form and numerical methods), interval computations and heuristic based methods.

Constrained Fuzzy Computations

Buckley and Eslami (2002) employ constrained fuzzy arithmetic operations to overcome the convergence problem in fuzzy Markov chains. They define crisp transition matrices where values are restricted by a closed and bounded domain, determined by $\alpha$-cuts, and the sum of all rows equal to one. The constraint of the transition matrix can be described by the following equation

$$C = \left\{ \bar{s}_{i,n_i}' = (s_{1,n_1}, s_{2,n_2}, \ldots, s_{q,n_q}) : s_{i,n_i} \geq 0, \sum_{i=1}^{q} s_{i,n_i} = 1 \right\},$$  \hspace{1cm} (7)

And the $\alpha$-cut domain can be defined by

$$\overline{\text{Dom}}_i(\alpha) = \left[ \prod_{j=1}^{q} \tilde{s}_{ij}(\alpha) \right] \cap C, \quad \text{Dom}(\alpha) = \left[ \prod_{i=1}^{q} \overline{\text{Dom}}_i(\alpha) \right], \quad 0 \leq \alpha \leq 1; \quad 1 \leq i \leq q$$  \hspace{1cm} (8)

The fuzzy steady state probabilities $(\tilde{s}_{i,n_i}(\alpha))$ are derived by a specific function of the $\alpha$-cut domain which bounds can be expressed as follows

$$\tilde{s}_{i,n_i}(\alpha) = \min \left\{ f_{ij}(s_{i,n_i}) : s_{i,n_i} \in \text{Dom}(\alpha) \right\}, \quad \overline{s}_{i,n_i}(\alpha) = \max \left\{ f_{ij}(s_{i,n_i}) : s_{i,n_i} \in \overline{\text{Dom}}(\alpha) \right\}$$  \hspace{1cm} (9)

For simple cases Buckley and Eslami (2002) suggest closed form solutions for Eq. (8) and (9), however, for more complex problems they suggest employing directed search algorithms. In this regard, their
approaches are not practical, because infinite number of trials can be driven from a FFLS (see also Dehghan et al., 2006). Huang and Tzeng (2007) propose a similar method where crisp values, representing the transition probabilities, are randomly chosen from the domain which makes the sum of all the columns equal to one. They also suggest using heuristic methods to approximate the bounds of the steady state probabilities.

**Interval Computations**

The aim of this group of methods is to extend the concept of inverse of a matrix with fuzzy numbers. There is a strong connection between interval and fuzzy matrices, because each \( \alpha \)-cut of a fuzzy matrix of fuzzy numbers \( [A]_\alpha \) is an interval matrix \( ([A], \overline{A}) \). Here we introduce Rohn’s (1993) approach for inverting interval matrices. This method considers a real matrix \( A \), which is derived from a fuzzy matrix \( \hat{A} \) and named a scenario, and approximates the fuzzy inverse with respect to scenarios. In this respect it is similar to the method of Buckley and Eslami (2002). Note that, for a satisfactory result \( \hat{A} \) should be regular.

Based on Rohn’s (1993) definition, the inverse of \( \hat{A} \) can be formulated as the narrowest interval matrix containing the set \( [A^{-1}]_\alpha \subset [\overline{A}^{-1}]_\alpha \) for \( \alpha \in [0, 1] \). Accordingly, the bounds of the inverse interval matrix \( B(\alpha) = [\underline{B}(\alpha), \overline{B}(\alpha)] \) can be stated by the following equations

\[
B_{ij}(\alpha) = \min\{(A^{-1})_{ij} | A \in [\overline{A}]_\alpha \} \quad \text{and} \quad \overline{B}_{ij}(\alpha) = \max\{(A^{-1})_{ij} | A \in [\overline{A}]_\alpha \}\quad (10)
\]

where \( i, j = 1, \ldots, q \) and \( \alpha \in [0, 1] \). To compute these bounds Rohn suggest using the centre matrix \( (1/2 \cdot (\max[A]_0 + \min[A]_0)) \) and radius matrix \( (1/2 \cdot (\max[A]_0 - \min[A]_0)) \) instead of the matrices \( A(\alpha) \) and \( \overline{A}(\alpha) \). For more detail about the computations the reader should refer to Rohn (1993) and Dehghan et al. (2006). We can conclude that inversion of an interval matrix is very time consuming when the matrix is large.

**Heuristic Based Methods**

There are many heuristic based methods in the literature for finding approximated solutions of a FFLS. However, as explained above, many of these methods do not deliver satisfactory non-negative solutions. The method proposed in this paper is based on splitting the fuzzy matrix \( [\hat{A}]_\alpha \) into two crisp matrices denoted by \( L = A(\alpha) \) and \( U = \overline{A}(\alpha) \). Assuming that \( L \) and \( U \) are nonsingular crisp matrices, we can write the following linear systems

\[
\begin{align*}
A(\alpha) \cdot l_{in}(\alpha) &= e(\alpha) \quad \Rightarrow \quad l_{in}(\alpha) = A(\alpha)^{-1} \cdot e(\alpha) \quad (11) \\
\overline{A}(\alpha) \cdot u_{in}(\alpha) &= \bar{e}(\alpha) \quad \Rightarrow \quad u_{in}(\alpha) = \overline{A}(\alpha)^{-1} \cdot \bar{e}(\alpha) \quad (12)
\end{align*}
\]

So we easily get the approximation of the \( \alpha \)-cuts of the fuzzy global priority vector \( [\underline{s}_{in}(\alpha), \overline{s}_{in}(\alpha)] \)

\[
\begin{align*}
\underline{s}_{in}(\alpha) = \left[ \min\{l_{in}(\alpha), u_{in}(\alpha)\}, \max\{l_{in}(\alpha), u_{in}(\alpha)\} \right] \quad (13)
\end{align*}
\]

for \( \alpha \in [0, 1] \). This method suggests solutions to an arbitrary FFLS and provides practical results.

**2.2.2. Numerical Example**

Let a stochastic fuzzy supermatrix be formed as:

\[
\mathbf{S} = \begin{bmatrix}
(0, 0, 0) & (0, 0, 0) & (0.667, 0.75, 0.8) & (0.1, 0.111, 0.125) \\
(0.143, 0.167, 0.2) & (0.667, 0.75, 0.8) & (0, 0, 0) & (0, 0) \\
(0.8, 0.833, 0.857) & (0.2, 0.25, 0.333) & (0, 0, 0) & (0, 0)
\end{bmatrix}
\]
Results obtained from the above discussed computational methods are summarized in Table 1.

### Table 1. Global Priority Values

<table>
<thead>
<tr>
<th>Decision Elements</th>
<th>Crisp ANP</th>
<th>Crisp SM</th>
<th>Fuzzy ANP</th>
<th>Rohn’s Inverse Method $\lambda = 0.5^*$</th>
<th>Rohn’s Inverse Method $\alpha$-cuts = 0</th>
<th>Rohn’s Inverse Method $\alpha$-cuts = 1</th>
<th>Heuristic Method $\alpha$-cuts = 0</th>
<th>Heuristic Method $\alpha$-cuts = 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.2151</td>
<td>0.2128</td>
<td>[0.1803, 0.2428]</td>
<td>[0.2150, 0.2151]</td>
<td>[0.1803, 0.2427]</td>
<td>[0.2150, 0.2150]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.2849</td>
<td>0.2872</td>
<td>[0.2633, 0.3157]</td>
<td>[0.2849, 0.2850]</td>
<td>[0.2779, 0.3010]</td>
<td>[0.2850, 0.2850]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.2495</td>
<td>0.2490</td>
<td>[0.2221, 0.2761]</td>
<td>[0.2496, 0.2497]</td>
<td>[0.2265, 0.2708]</td>
<td>[0.2496, 0.2496]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.2505</td>
<td>0.2510</td>
<td>[0.2085, 0.2922]</td>
<td>[0.2503, 0.2504]</td>
<td>[0.2085, 0.2922]</td>
<td>[0.2504, 0.2504]</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

* Values are defuzzified using the total integral value method (Liou and Wang, 1992)

### 3. Conclusion

On the basis of the experiments we can conclude that all introduced methods have delivered satisfactory solutions. The difference between the crisp ANP and fuzzy ANP with crisp supermatrix arises from the use of non-symmetric fuzzy numbers. The most time consuming method is Rohn’s approach while delivering the most robust results. In this study we suggest a Markov based approach for calculating the fuzzy limit matrix in ANP, accompanied by several alternative computational methods that allow computing the steady state distribution of the fuzzy supermatrix and, thereby for considering cumulative uncertainty.

### 4. References


**Scope/Topic: Multi-Criteria Decision Making and Decision Analysis**