ABSTRACT

In this paper, we deal with the problem of joint maintenance and production under a subcontracting and withdrawal right constraints. The manufacturing system consists of a machine $M$ that produces a single product in order to satisfy a random demand under given service level constraint. We first establish an optimal production plan which minimizes the total inventory and production cost taking into consideration the subcontractor and the withdrawal right constraints. Secondly, using this optimal production plan we derive an optimal maintenance plan which minimizes the total maintenance cost and takes into account the influence of production rates on the machine degradation. Finally, a numerical example is studied in order to apply the developed approach.

Keywords: Random demand, service level, stochastic model, withdrawal right, subcontracting, maintenance strategy, production policy.

1. INTRODUCTION

The combination maintenance/production plans is one of the problem studied by many researches. In deed the development of an economically production and optimal maintenance plans which minimizes the total cost including production, inventory and maintenance is among the important actions of a hierarchical decision manufacture process.

The production/maintenance planning provided by the stochastic optimal control problem is complex to solve. The difficulty owing to factors resembling to a variety of uncertainties. These uncertainties are usually related to the randomness of the demand, the difficulty of predicting it throughout future periods, the material availability and Failures rate variation. Some studies based on the stochastic optimal control problem. (Silva Filho, and Cezarino 2004) showed that it is possible to extend the unconstrained HMMS model in order to deal with a chance-constrained stochastic production planning problem under the hypothesis of imperfect inventory information variables. The HMMS model is usually applied as a benchmarking tool for comparing different aggregate production planning approaches, and, as well to provide managers with insights about the use of the company’s material resources. The linear decision rule developed by Holt, Modigliani, Muth and Simon-HMMS-(1960) can be considered as an important contribution for strategic production planning decision.

In the context of integrated approach maintenance and production control, a number of studies deal with the issue of the importance of the choice of the production/maintenance policies for the minimization of the total cost. (Buzacott 1967) is among the first authors who dealt with the problem of integrated production-maintenance strategies. He studied the role of buffer stocks in increasing the system productivity. Rezg and al. (2008) presented a mathematical model and a numerical procedure which allows determining a joint optimal inventory control and age based preventive maintenance policy for a randomly failing production system.

Increasingly competitive global industrial world, the relationships between enterprises is getting improved towards more cooperation and collaboration. In this context, many companies have recourse to the industrial subcontracting which became a very widespread practice to face competition.

Recently, in the context of an integrated maintenance policy for a manufacturing system calling upon several subcontractors, (Dellagi et al. 2007) developed a
maintenance strategy integrating a subcontracting constraint. They considered a production system represented by a machine producing a single product type to satisfy a constant demand. The machine calls upon sub-contracting represented by a second machine to complete the entire demand exceeding the maximum machine capacity. Two preventive maintenance policies for the contractor machine were studied: the first one does not take into account the state of the subcontracting machine whereas the second one does. In this paper we consider a randomly failing manufacturing system which has to satisfy a random demand. We develop a stochastic dynamic model in order to establish sequentially the economically production plan and the optimal preventive maintenance periods, take into account the influence of withdrawal right (where the products returned by the customer who are still new) and the subcontractor has the role of recycling and remanufacturing the not conformed products reject by the customer.

The remainder of the paper is organized as follows.

Section 2 states the problem setting, section 4 formulates and developed the production problem and section 5 defines and developed the equations of maintenance policy. In Section 6 we present a simple numerical example in order to illustrate the analytical results. Finally, Section 7 provides the conclusions.

2. PROBLEM SETTING

The manufacturing system, studied, is composed of a machine $M$ which produces a single product in order to meet the random demand $d$ law characterized by a Normal distribution facing the system at a minimum cost.

The production system consists of two stores and random demand is satisfied from the first store, where the manufactured products are stored. The part of returned products that are still good state and packaged are collected in the first store, called the withdrawal right. The other parts of not conformed products are collected in the second store and then remanufactured using subcontractor machine.

The machine $M$ is subject to a random failure. The probability degradation law of machine $M$ is described by the probability density function of time to failure $f(t)$ and for which the failure rate $\lambda(t)$ increases with time and according to the production rate $u(t)$. Failures of machine $M$ can be prevented by a preventive maintenance action which is scheduled according to its history.

The first objective is to minimize the sum of the quadratic cost from described inventory levels in two stores and from described manufacturing and remanufacturing. Secondly, using the production plan obtained by the production policy, we determined the optimal preventive maintenance periods. The use of the optimal production plan in maintenance study is justified by the influence of the production rate on the linear degradation of the manufacturing system.

3. NOTATION

$H, \Delta t$: finite production horizon
$\Delta t$: period length of production
$u(k)$: production rate by machine $M_1$ during period $k$ $(k=0,1,\ldots,H)$
$d_k$ : demand quantity during period $k$ $(k=0,1,\ldots,H)$
$V_{d_k}$ : demand variance during period $k$ $(k=0,1,\ldots,H)$
$\bar{d}_k$: average demand during period $k$ $(k=0,1,\ldots,H)$
$S_k$: inventory level at the end of period $k$ $(k=0,1,\ldots,H)$
$C_{p_M}$: unit production cost of machine $M_1$
$C_{p_S}$: unit production cost of subcontractor machine $M_S$
$C_S$: Inventory holding cost of product unit during one period
$\Gamma_p$: maintenance cost
$M_p$: preventive maintenance action cost
$M_s$: corrective maintenance action cost
$mu$: monetary unit
$U_{max}$: maximal production rate of machine $M$
$\theta$: probabilistic index (related to customer satisfaction and expressing the service level)
$R(t)$: reliability function
$f(t)$: probability density function associated with the time to failure of $M$
$\lambda_0(t)$: nominal failure rate corresponding to the maximal production rate.
$\lambda_k(t)$: Machine failure rate function during period $k$ $(k=0,1,\ldots,H)$
$\beta_0$: machine $M_1$ availability rate
$S_0$: initial inventory.

4. PRODUCTION POLICY

4.1 Production problem formulation
It is assumed that the horizon is partitioned equally into \( H \) periods of length \( H/\Delta t \). Let \( \{f_{i,k}, k = 1, \ldots, N\} \) represent holding and production costs and \( E[f] \) denotes the mathematical expectation operator. The following stochastic linear program provides an optimal production plan over the planning horizon:

\[
f(u) = C_s \times \left( E[S_1(\tau)]^2 + E[S_2(\tau)]^2 \right) + \sum_{k=1}^{H-1} \left[ C_p \times u(k)^2 + C_p \times \hat{u}(k)^2 \right] + \sum_{k=1}^{H-1} \left[ C_p \times u(k)^2 + C_p \times \hat{u}(k)^2 \right] \text{ with } k \in \{0,1,\ldots,H-1\}.
\] (1)

Subject to:

\[
S_1(k+1) = S_1(k) + u_1(k) + \mu \cdot k \cdot d(k-\tau) - \delta \cdot d(k) \quad k = 0,1,\ldots,H-1
\] (2)

\[
\text{Prob}[S_1(k+1) \geq \theta] \quad k = 0,1,\ldots,H-1
\] (3)

\[
S_2(k+1) = S_2(k) + \left(1 - \mu \right) \cdot \delta \cdot d(k-\tau) - u_2(k) \quad k = 0,1,\ldots,H-1
\] (4)

\[
u_1(k) = \beta_1 \cdot u_2(k) \quad 0 < \beta_1 < 1
\] (5)

\[
0 \leq u(k) \leq U_{\max} \quad k = 0,1,\ldots,H-1
\] (6)

The constraint (2) denotes the inventory balance equation for the principle store. Note also that \( d(k) \) denotes the uncertainty about the fluctuation of demand. The service level requirement constraint for each period is expressed by the constraint 3. The inventory balance equation for the second store described by the constraint (4). The relation (5) defines the subcontractor production rate according to their availability \( \beta_1 \). Finally, the last constraint defines an upper bound on the production level during each period \( k \) cannot exceed a given maximal production rate \( U_{\max} \).

4.2 Analytical study

In this section, we determine a deterministic production formulation in order to easier the resolution of our stochastic problem.

- Production inventory costs:

The first step is converted the objective production function to deterministic equivalent pattern as follows:

**Lemma1:**

\[
f(u) = \sum_{k=0}^{H} C_s \times \left[ \hat{S}_1(k)^2 + \hat{S}_2(k)^2 + k \cdot \delta^2 \cdot \left(1 - \mu\right)^2 \cdot \sigma_d^2 \right] + \sum_{k=1}^{H-1} \left[ C_p \times \hat{u}(k)^2 + C_p \times \hat{u}(k)^2 \right]
\] (7)

- Service level constraint:

To continue transforming the stochastic problem into an equivalent deterministic one, we consider a service level constraint in a deterministic form by specifying through the following lemma a minimum cumulative production quantity depending on the service level requirements.

**Lemma 2:**

We recall that \( \theta \) defines the service level constraint. This constraint is expressed as follows:

\[
\text{Prob}[S_1(k+1) \geq \theta] \geq \alpha \text{ with } 0 \leq u(k) \leq U_{\max}
\] (8)

Then, for \( k = 0,1,\ldots,H-1 \) we have:

\[
\text{Prob}[S_1(k+1) \geq \theta] \geq u(k) \geq U_{\max}
\] (9)

Where \( U_{\max} \) represents a minimum cumulative production quantity expressed as follows:

\[
U_{\max} = U_{\max}\left[S_1(k), \theta\right] = 0,1,\ldots,H-1
\]

**5. MAINTENANCE POLICY**

5.1 Maintenance problem formulation

The maintenance policy characterized by the cost associated with a given schedule of future preventive maintenance and replacement activities. The joint optimization strategy considers these costs, based on optimal production rates previously found by the production policy, in order to determine the optimal maintenance strategy characterized by the optimal partition number \( N^* \) of preventive maintenance actions and the length of the partition \( T^* \).

The maintenance policy adopted is a preventive systematic maintenance block type as follows:

Make a preventive maintenance action on machine \( M \) every \( T \) time units with \( T > \Delta t \) and perform a corrective maintenance on machine \( M \) in case of failure between consecutive preventive maintenance actions.
The preventive maintenance actions are assumed to be perfect, the machine is then considered in a state as good as new after each maintenance action. However, when a failure occurs between successive preventive maintenance actions, a minimal repair action is performed to maintain the system during the current period. As a consequence, the failure rate is undisturbed. It is also assumed that the corrective and preventive maintenance actions durations are negligible. The analytic expression of the total maintenance cost is as follows, with \( T = H/N, \ N \in \{1,2,3,.....\} \).

\[
\Gamma_p(U,N) = (N-1) \cdot M_p + M_c \cdot \varphi_p(U,N)
\]

(9)

Where \( \varphi_p(U,N) \) corresponds to the expected number of failure, i.e. the average number of failures that can occur during the horizon \( H \), considering the production rate variation for each production period \( k \) for \([k\Delta t, (k+1)\Delta t] \) with \( k \in [0,H-1] \).

It’s obvious that the maintenance policy is tightly related to the system degradation. That is why, the maintenance plan should be optimized considering the production plan previously established for the \( H \) periods of the planning horizon, in order to take into account the influence of the production rate on the linear failure rate \( \lambda(t) \).

5.2 Analytical study

For this maintenance policy, we assumed that machine degradation varies linearly according to the production rate. Each period \( k \) of the horizon \( H \Delta t \) is characterized by its own production rate \( \mu(k) \) established from the production plan. The linear failure rate evolves in each interval according to the production rate adopted in that period. It also depends on the failure rate cumulated at the end of the previous period.

In fact, the failure rate in the interval \( k \) is expressed as follows:

\[
\lambda_k(t) = \lambda_{n,k} \Delta t + \frac{U_k}{U_{max}} \lambda_k(t) \quad \lambda_0(0) = \lambda_0 \text{ and } \Delta \lambda_k(t) = \frac{U_k}{U_{max}} \lambda_k(t)
\]

(10)

\( \lambda_n(t) \) is the nominal failure rate corresponding to the maximal production rate. We can write the failure rate function as expressed in the following way by (Hajej et al. 2009):

\[
\lambda_k(t) = \lambda_0 + \sum_{j=1}^{k} \frac{U_j}{U_{max}} \lambda_n(\Delta t) + \frac{U_k}{U_{max}} \lambda_n(t) \text{ with } t \in [0,\Delta t]
\]

(11)

The average number of failures over the horizon \( H \Delta t \) is expressed as follows:

\[
\varphi_p(U,N) = \sum_{j=0}^{N-1} \left[ \left( (j+1)T - \frac{U_k}{U_{max}} \right) \int_0^{(j+1)T} \lambda_k(t) dt + \int_0^{(j+1)T} \frac{U_k}{U_{max}} \lambda_n(t) dt \right]
\]

(12)

With \( T = H/N, \ \text{In} \ (T/\Delta t) \): integer part of \( T/\Delta t \)

6. NUMERICAL EXAMPLE

The following numerical example is considered to illustrate our approach, we consider a company represented by machine \( M \) which has to satisfy a stochastic demand assumed Gaussian over a finite horizon \( H \Delta t \), with a mean \( \bar{D}_k \) and a variance \( V_{dk} \).

The number \( H \) of periods \( \Delta t \) is equal to 72, with \( \Delta t = 1 \). The machine \( M \) has a degradation law characterized by a Weibull distribution. The Weibull scale and shape parameters are \( \beta = 100 \) and \( \alpha = 2 \) while \( M_s = 3000 \) mu and \( M_p = 500 \) mu. The only information known about \( M_s \) is the availability rate \( \beta_s = 0.9 \).

The following data are used for the other parameters: \( C_{ps} = 2 \mu \), \( C_{ps} = 6 \mu \), \( U_{max} = 11 \), service level \( \theta = 0.9 \), \( C_s = 3 \mu \), initial inventory \( S_0 = 15 \). The variance \( V_{dk} = 1.2 \). The rate of returned products \( \delta \in [0,0.5] \) and \( \tau = 1 \). The average demand is presented in table 1 below.

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Table 1: Mean demand

Applying the numerical procedure we obtained the economically production plan and the optimal maintenance period.
According to previous figures, we notice the influence of product returns in terms of production plan and the optimal preventive maintenance period.

Figure 2 shows the production plans of different values of $\delta$ ($\delta=0.03$, $\delta=0.4$). When $\delta$ increases, the production plan decreases, this is logical because we have more products returned by the subcontractor to the stock.

Figure 3 shows the total maintenance cost, $\gamma_p(U,N)$, according to $N$. The optimal number of partitions $N^*$ is:

For $\delta(\alpha_1)=0.03$: $N^*=3$, $\gamma^*_p=313669.6\mu$ and the optimal period of maintenance $T^*=H/N^*=24\Delta t$

For $\delta(\alpha_2)=0.4$: $N^*=2$, $\gamma^*_p=313582.8\mu$ and the optimal period of maintenance $T^*=H/N^*=36\Delta t$

In the same way as the previous comment about the production plan, the period of maintenance has become increasingly important when $\delta$ value increases. This is logical $\delta$ grows, so that $u$ is decreasing, thus the preventive maintenance actions are less.

7. Conclusion

In this paper, we studied a problem of jointly production/maintenance policy for a manufacturing system subjected to a random failure calling upon a subcontractor machine in order to satisfy a random demand. Points of view reliability, a minimal repair is practiced at every failure. In order to reduce the failure frequency, preventive maintenance actions is scheduled according to the production rate.

Firstly, given a randomly demand and a satisfaction customer rate, we have formulated and solved a stochastic production problem in order to obtain an economically production plan.

Secondly, using the optimal production plan in the maintenance problem formulation, we established an economical maintenance scheduling, in the case of the linear degradation of machine, taking into consideration the influence of the production plan on the manufacturing system deterioration.

REFERENCES


Proof of lemma 2:

\[ S_i(k+1) = S_i(k) + u_i(k) + u(k) + \mu \cdot \delta \cdot d(k - \tau_i) - d(k) \]

\[ \text{Prob} \left( S(k+1) \geq 0 \right) \geq \theta \]

\[ \text{Prob} \left( S_i(k) + u_i(k) + u(k) + \mu \cdot \delta \cdot d(k - \tau_i) - d(k) \geq 0 \right) \geq \theta \]

\[ \text{Prob} \left( S_i(k) + u_i(k) + u(k) + \mu \cdot \delta \cdot d(k - \tau_i) - \hat{d}(k) \geq d(k) \right) \geq \theta \]

\[ \text{Prob} \left( d(k) - \hat{d}(k) \leq S_i(k) + u_i(k) + u(k) + \mu \cdot \delta \cdot d(k - \tau_i) - \hat{d}(k) \right) \geq \theta \]

\[ \left( \frac{d(k) - \hat{d}(k) - \mu \cdot \delta \cdot d(k - \tau_i) + \mu \cdot \delta \cdot \hat{d}(k - \tau_i)}{V_d \times V_{d,k - \tau_i}} \right) \leq \text{Prob} \leq \left( \frac{S_i(k) + u_i(k) + u(k) - \hat{d}(k) + \mu \cdot \delta \cdot \hat{d}(k - \tau_i)}{V_d \times V_{d,k - \tau_i}} \right) \geq \theta \]

\[ \left( \frac{1 - \frac{d(k) - \hat{d}(k)}{V_{d,k - \tau_i}} - \frac{d(k - \tau_i) - \hat{d}(k - \tau_i)}{V_d}}{V_d} \right) \leq \text{Prob} \leq \left( \frac{S_i(k) + u_i(k) + u(k) - \hat{d}(k) + \mu \cdot \delta \cdot \hat{d}(k - \tau_i)}{V_d \times V_{d,k - \tau_i}} \right) \geq \theta \]

Note that \( X = \left( \frac{d(k) - \hat{d}(k)}{V_{d,k}} \right) \) : being a Gaussian random variable with an identical distribution as \( d(k) \).

and \( Y = \left( \frac{d(k - \tau) - \hat{d}(k - \tau)}{V_{d,k - \tau}} \right) \) : being a Gaussian random variable with an identical distribution as \( d(k - \tau) \)

This formulation is equivalent to

\[ \text{Prob} \left( A \times X + B \times Y \leq C \right) \geq \theta \quad \text{with} \quad A = \frac{1}{V_{d,k - \tau}} \]

and \( B = -\frac{\mu \cdot \delta}{V_d} \)

\( X = A \times \hat{X} \) is a Gaussian random variable with an identical distribution as \( f_{\hat{X}} = \frac{1}{A} \times f_X \), with mean an identical distribution as 

\[ \hat{A} \times \hat{X} = \left( \frac{1}{V_{d,k - \tau}} \right) \times \hat{X} \]

And \( Y = B \times \hat{Y} \) is a Gaussian random variable with an identical distribution as \( f_Y = -\frac{1}{B} \times f_{\hat{Y}} \), with mean 

\[ B \times \hat{Y} = \left( -\frac{\delta}{V_d} \right) \times \hat{Y} \]

Thus \( T = X + Y \) is a Gaussian random variable with an identical distribution as \( \hat{X} \times \hat{X} \times f_{\hat{Y}} \), with mean 

\[ A \times \hat{X} + B \times \hat{Y} \geq 0 \]

and \( \phi \) is a cumulative Gaussian distribution function of

\[ T' = \phi \left( \frac{S_i(k) + u_i(k) + u(k) - \hat{d}(k) + \mu \cdot \delta \cdot \hat{d}(k - \tau_i)}{V_d \times V_{d,k - \tau}} \right) \geq \theta \]

Since \( \lim_{d_i \to -\infty} \phi_{d_i} = 0 \) and \( \lim_{d_i \to +\infty} \phi_{d_i} = 1 \), the function \( \phi_{d_i} \) is strictly increasing, and we note that is indefinitely differentiable. That’s why we conclude that \( \phi_{d_i} \) is invertible.

Thus,

\[ \frac{S_i(k) + u_i(k) + u(k) - \hat{d}(k) + \mu \cdot \delta \cdot \hat{d}(k - \tau_i)}{V_d \times V_{d,k - \tau}} \geq \phi^{-1} \left( \theta \right) \]

Thus it can consequently be concluded that

\[ \text{Prob} \left[ S_i(k+1) \geq 0 \right] \geq \theta \Rightarrow u(k) \geq U_i \left( \hat{S}_i(k) \right) \quad k = 0, 1, \ldots, H - 1 \]

\[ U_i \left( \hat{S}_i(k) \right) = \left[ V_{d,k} \times V_{d,k - \tau} \right] \times \phi^{-1} \left( \theta \right) + \hat{d}(k) - \delta \cdot \hat{d}(k - \tau_i) - \hat{S}_i(k) \quad k = 0, 1, \ldots, H - 1 \]