A TANDEM QUEUE WITH DEPENDENT STRUCTURE FOR INTERARRIVAL AND SERVICE TIMES: AN ESTIMATION STUDY

B.Chandrasekar¹, P.Chandrasekhar¹, N.Ramesh Kumar¹, VSS Yadavalli²

¹Department of Statistics, Loyola College, Chennai- 600 034, India
²Department of Industrial and Systems Engineering, University of Pretoria, 0002, South Africa
(sarma.yadavalli@up.ac.za)

Abstract
Consider a single channel two station tandem queue with zero queue capacity and with blocking (Taha, 1989). Assuming that the arrival process and the service process are dependent, we obtain the transient solution for the system when the joint distribution of interarrival time, service time at Station 1 and service time at Station 2 is trivariate exponential. Also, CAN estimators and asymptotic confidence limits for the expected service time per customer in the system are obtained.

Keywords
CAN estimator - expected service time - multivariate central limit theorem - Slutsky theorem - tandem queue - trivariate exponential distribution.

Introduction
Parametric estimation, Interval estimation and Bayesian estimation are some of the essential tools to understand any random phenomena using stochastic models. Analysis of queueing systems in this direction has not received due attention. Whenever the systems are fully observable in terms of their basic random components such as inter arrival times and service times, standard parametric techniques of statistical theory are quite appropriate. Recently, Bhat (2003) has provided an overview of methods available for estimation, when the information is restricted to the number of customers in the system at some discrete points in time. Bhat (2003) has also described how maximum likelihood estimation is applied directly to the Markov chain in the queue length process in M/G/1 and GI/M/1 queues. Yadavalli et al (2004) have obtained asymptotic confidence limits for the expected waiting time per customer in the queues of M/M/1/∞ and M/M/1/N. Further, Yadavalli et al (2006) have extended the same results to c parallel servers (c≥1). Chandrasekhar et al (2008) have obtained asymptotic confidence limits for the expected service time per customer in the system in a two station tandem queue with zero queue capacity and with blocking. They have assumed that service times at each station are exponential random variables with the same service rate. Further, Chandrasekhar et al (2006) have studied in detail a two station tandem queue with dependent structure for service times wherein they have assumed that the joint distribution of service times at Station 1 and Station 2 is bivariate exponential and obtained asymptotic confidence limits for the average number of customers in the system and the expected service time per customer in the system.

Generally speaking, the queueing models assume that each service channel consists of only one station. Situations do exist where each service channel may consist of several stations in series. In this situation, a customer must pass through successively all the stations before completing his service. Such situations are known as queues in series or tandem queues. e.g., (a) In a manufacturing process, units must pass through a series of service channels (work stations), where each service channel performs a given task or job. (b) In a university registration process, each registrant must pass through a series of counters such as advisor, department chairman, cashier etc. (c) In a clinical physical examination procedure, a patient goes through a series of
stages such as lab tests, ECG, chest X-ray etc. In all these model structures, it is not only sufficient to know how many persons are there in the system but also where they are. An attempt is made in this chapter to study in detail a twostation tandem queue with blocking (Taha, 1989). Moment and CAN estimators and asymptotic confidence limits for the expected service time per customer in the system are obtained under the assumption that the joint distribution of interarrival time, service times at Station 1 and Station 2 is trivariate exponential. In the following section, system description and assumptions are given briefly.

System Description and Assumptions

Consider a simplified one-channel queueing system consisting of two series stations as below:

![System configuration](image)

A customer arriving for service must go through Station 1 and Station 2 before completing his service. The assumptions of the model are as follows:

1. Arrivals occur according to a Poisson process with rate $\lambda$.
2. Let $T_1$ and $T_2$ denote the service times of station 1 and station 2 respectively and $R$ denote the interarrival time. Assuming that the two service stations are identical, it is only appropriate to consider the following trivariate exponential distribution for $T_1$, $T_2$ and $R$ with the survival function (Marshall and Olkin, 1967) given by

$$F(t_1, t_2, t_3) = e^{-\lambda_1(t_1+t_2)+\lambda_1t_3-[\max(t_1,t_3)+\max(t_2,t_3)]}, \ t_1, t_2, t_3 > 0, \ \lambda_1, \lambda_2 > 0, \ \lambda_3 \geq 0.$$ (2.1)

3. No queues are permitted in front of Station 1 or Station 2.
4. Each station is either free or busy.
5. Station 1 is said to be blocked, if the customer in Station 1 completes his service before station 2 becomes free. In this case, he cannot wait between the stations since this is not allowed and the customer remain in Station 1 itself.

Note: It is easy to observe that

(i) $T_1$ and $T_2$ are independent and identically distributed exponential random variables each with the parameter $(\lambda_1+\lambda_3)$.

(ii) $R$ is exponential with the parameter $(\lambda_2+2\lambda_3)$, not necessarily independent of $(T_1,T_2)$

(iii) $(T_i,R)$ is bivariate exponential, $i=1,2$

(iv) $E(T_i) = E(T_2) = \frac{1}{\lambda_i + \lambda_3}$ and $E(R) = \frac{1}{\lambda_2 + 2\lambda_3}$

(v) The random variables $T_1$, $T_2$ and $R$ are independent if and only if $\lambda_3=0$.

Analysis of the System

Let the symbols 0, 1 and b represent free, busy and blocked states of a station. Let $X(t)$ and $Y(t)$ respectively denote the states of Station 1 and Station 2 and the vector process $Z(t) = \{(X(t), Y(t)), t \geq 0\}$ with state space

$$E = \{(0,0),(0,1),(1,0),(1,1),(b,1)\}$$ (1)

denote the state of the system at time $t$. 

Let \( p_{ij}(t) = \Pr\{Z(t) = (i, j)\}, \forall (i, j) \in E \) represent the probability that the system is in state \((i, j)\) at time \( t \) with the initial condition \( p_{00}(0)=1\). From the infinitesimal generator of the process given in (3.2), we have the following system of differential difference equations:

\[
\begin{align*}
P_{00}(t) &= -\left(\lambda_2 + 2\lambda_3\right) p_{00}(t) + \left(\lambda_1 + \lambda_3\right) p_{01}(t) \\
\lambda_1 P_{01}(t) &= -\left(\lambda_1 + \lambda_2 + 3\lambda_3\right) p_{01}(t) + \left(\lambda_1 + \lambda_3\right) p_{10}(t) + \left(\lambda_1 + \lambda_3\right) p_{b1}(t) \\
\lambda_2 P_{10}(t) &= \left(\lambda_2 + 2\lambda_3\right) p_{00}(t) - \left(\lambda_1 + \lambda_3\right) p_{10}(t) + \left(\lambda_1 + \lambda_3\right) p_{11}(t) \\
\lambda_1 P_{11}(t) &= \left(\lambda_2 + 2\lambda_3\right) p_{01}(t) - 2\left(\lambda_1 + \lambda_3\right) p_{11}(t) \\
\lambda_1 P_{b1}(t) &= \left(\lambda_1 + \lambda_3\right) p_{11}(t) - \left(\lambda_1 + \lambda_3\right) p_{b1}(t)
\end{align*}
\]

**Steady State Solution**

The equations (2)-(6) can be solved using the fact that \( \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} p_{ij}(t) = 1 \). Since we wish to study the stationary behaviour of the system, let \( \lim_{t \to \infty} p_{ij}(t) = p_{ij} \).

\[
\begin{align*}
P_{00} &= \frac{A_1}{D}, \quad P_{01} = \frac{A_2}{D}, \quad P_{10} = \frac{A_3}{D}, \quad P_{11} = \frac{A_4}{D}, \quad P_{b1} = \frac{A_5}{D}
\end{align*}
\]

Where, \( A_1 = 2(\lambda_1 + \lambda_3); \quad A_2 = 2(\lambda_1 + \lambda_3)(\lambda_2 + 2\lambda_3); \quad A_3 = (\lambda_2 + 2\lambda_3)[(\lambda_2 + 2\lambda_3) + 2(\lambda_1 + \lambda_3)] \)

\[
A_4 = (\lambda_2 + 2\lambda_3)^2; \quad D = [2(\lambda_1 + \lambda_3)^2 + 4(\lambda_1 + \lambda_3)(\lambda_2 + 2\lambda_3) + 3(\lambda_2 + 2\lambda_3)^2]
\]

**Particular case**

When \( \lambda_1 = \mu, \lambda_2 = \lambda, \lambda_3 = 0 \), our model reduces to that of Taha, 1989. Also our solution is in agreement with the solution of the model of Taha, 1989 with \( \lambda_1 = \mu, \lambda_2 = \lambda \) and \( \lambda_3 = 0 \).

**Performance measures**

The expected number of customers in the system is given by

\[
L_s = (p_{01} + p_{10}) + 2(p_{11} + p_{b1})
\]

\[
= \frac{(\lambda_2 + 2\lambda_3)[4(\lambda_1 + \lambda_3) + 5(\lambda_2 + 2\lambda_3)]}{[2(\lambda_1 + \lambda_3)^2 + 4(\lambda_1 + \lambda_3)(\lambda_2 + 2\lambda_3) + 3(\lambda_2 + 2\lambda_3)^2]}
\]

The probability that a customer will enter Station 1, observe that, \( W_s \) represents the expected service time per customer in the system, since no queues are allowed and is given by

\[
W_s = \frac{L_s}{\lambda_{\text{eff}}} = \frac{(4\lambda_1 + 5\lambda_2 + 14\lambda_3)}{2(\lambda_1 + \lambda_3)(\lambda_1 + \lambda_2 + 3\lambda_3)}
\]

In the next section, we obtain moment estimator and CAN estimator for the expected number of customers as well as the expected service time per customer in the system.

**Moment and CAN estimators for the expected number of customers and the expected service time per customer in the system**

**An Estimator Based On the Moments**

Let \((Y_{1i}, Y_{2i}, Y_{3i})\), \(i=1, 2, 3… n\) be a random sample of size \( n \) drawn from trivariate exponential service times and interarrival time population with the survival function given by (1). It is clear that \( \bar{Y}_1, \bar{Y}_2 \) and \( \bar{Y}_3 \) are the moment estimators of \( \frac{1}{\lambda_1 + \lambda_3}, \frac{1}{\lambda_1 + \lambda_3} \) and \( \frac{1}{\lambda_2 + 2\lambda_3} \) respectively.
where $\bar{Y}_1, \bar{Y}_2,$ and $\bar{Y}_3$ are the sample means of service times at Station 1, Station 2 and interarrival times respectively.

Let $\theta_1 = \frac{1}{\lambda_1 + \lambda_3}$ and $\theta_2 = \frac{1}{\lambda_2 + 2\lambda_3}$.

The average service time per customer in the system given in (9) reduces to

$$W_s = \frac{\theta_1(5\theta_1 + 4\theta_2)}{2(\theta_1 + \theta_2)}$$

and hence an estimator of $W_s$ based on moments is given by

$$\hat{W}_s = \frac{\bar{Y}_1(5\bar{Y}_1 + 4\bar{Y}_3)}{2(\bar{Y}_1 + \bar{Y}_3)}$$

It may be noted that $\hat{W}_s$ given in (11) are respectively real valued functions in $\bar{Y}_1$ and $\bar{Y}_3$, which are also differentiable.

**CAN Estimator**

By applying the multivariate central limit theorem, it readily follows that

$$\sqrt{n}[(\bar{Y}_1, \bar{Y}_2, \bar{Y}_3) - (\theta_1, \theta_2, \theta_3)] \xrightarrow{d} N_3(0, \Sigma)$$

as $n \to \infty$, where the dispersion matrix $\Sigma = (\sigma_{ij})$ is given by (Barlow and Proschan, 1975)

$$
\Sigma = \begin{bmatrix}
\theta_1^2 & 0 & \frac{\lambda_3\theta_1^2\theta_2^2}{\theta_1 + \theta_2 - \theta_1\theta_2\lambda_3} \\
0 & \theta_2^2 & \frac{\lambda_3\theta_1^2\theta_2^2}{\theta_1 + \theta_2 - \theta_1\theta_2\lambda_3} \\
\frac{\lambda_3\theta_1^2\theta_2^2}{\theta_1 + \theta_2 - \theta_1\theta_2\lambda_3} & \frac{\lambda_3\theta_1^2\theta_2^2}{\theta_1 + \theta_2 - \theta_1\theta_2\lambda_3} & \theta_2^2
\end{bmatrix}
$$

From Radhakrishna Rao (1974), we have

$$\sqrt{n}[(\hat{W}_s - W_s)] \xrightarrow{d} N(0, \sigma^2(\theta))$$

as $n \to \infty$, where $\theta = (\theta_1, \theta_2, \theta_3)$ and

$$\sigma^2(\theta) = 2\theta_1^2\left(\frac{\partial W_s}{\partial \theta_1}\right)^2 + \theta_2^2\left(\frac{\partial W_s}{\partial \theta_2}\right)^2 + \frac{4\lambda_3\theta_1^2\theta_2^2}{(\theta_1 + \theta_2 - \theta_1\theta_2\lambda_3)^2}\left(\frac{\partial W_s}{\partial \theta_1}\right)\left(\frac{\partial W_s}{\partial \theta_2}\right)$$

(12)

By substituting for partial derivatives $\left(\frac{\partial W_s}{\partial \theta_i}\right), i=1, 2$ in (20), we get $\sigma^2(\theta)$. Thus $\hat{W}_s$ is a CAN estimator of $W_s$ (An estimator $T_n$ is said to be CAN estimator of $g(\theta)$, if the asymptotic distribution of $\sqrt{n}(T_n - g(\theta))$ is normal, and,
There are several methods for generating CAN estimators and the Method of Moments and the Method of maximum likelihood are commonly used to generate such estimators (Sinha, 1986).

**Confidence Interval for the Expected Service Time**

Let $\hat{\sigma}^2(\hat{\theta})$ be the estimator of $\sigma^2(\theta)$ obtained by replacing $\theta$ by a consistent estimator $\hat{\theta}$ namely- $(\bar{X}, \bar{Y})$. Let $\sigma^2 = \sigma^2(\theta)$. Since $\sigma^2(\theta)$ is a continuous function of $\theta$, $\hat{\sigma}^2$ is a consistent estimator of $\sigma^2(\theta)$. (i.e), $\hat{\sigma}^2 \xrightarrow{p} \sigma^2(\theta)$ as $n \to \infty$. By Slutsky theorem, we have

$$\frac{\sqrt{n}(\hat{W}_n - W_0)}{\hat{\sigma}} \xrightarrow{p} N(0,1)$$

i.e, $\text{Pr} \left[ -k_\frac{\alpha}{2} < \frac{\sqrt{n}(\hat{W}_n - W_0)}{\hat{\sigma}} < k_\frac{\alpha}{2} \right] = (1 - \alpha)$ (13)

Where $k_\frac{\alpha}{2}$ is obtained from normal tables. Hence, a $100(1-\alpha)$ % asymptotic confidence interval $W_n$ is given by,

$$\hat{W}_n \pm k_\frac{\alpha}{2} \frac{\hat{\sigma}}{\sqrt{n}}$$

Where $\hat{\sigma}$ is obtained from (12). Similarly, one can obtain a $100(1-\alpha)$ % asymptotic confidence interval for $W_n$ from (10) and (12).

**Numerical Illustration**

The following table presents the waiting times for different parameters of $\lambda_1$, $\lambda_2$, and $\lambda_3$.

<table>
<thead>
<tr>
<th>$\lambda_1$</th>
<th>0.1</th>
<th>0.05</th>
<th>0.033333</th>
<th>0.025</th>
<th>0.02</th>
<th>0.016667</th>
<th>0.014286</th>
<th>0.0125</th>
<th>0.011111</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_2$</td>
<td>0.1</td>
<td>0.115</td>
<td>0.09225</td>
<td>0.085222</td>
<td>0.081813</td>
<td>0.0798</td>
<td>0.078472</td>
<td>0.077531</td>
<td>0.076828</td>
<td>0.076284</td>
</tr>
<tr>
<td>0.05</td>
<td>0.042</td>
<td>0.030375</td>
<td>0.026917</td>
<td>0.025266</td>
<td>0.0243</td>
<td>0.023667</td>
<td>0.023219</td>
<td>0.022887</td>
<td>0.02263</td>
<td>0.022425</td>
</tr>
<tr>
<td>0.025</td>
<td>0.0214844</td>
<td>0.016074</td>
<td>0.014178</td>
<td>0.013722</td>
<td>0.013401</td>
<td>0.013163</td>
<td>0.012979</td>
<td>0.012833</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.02</td>
<td>0.014844</td>
<td>0.011936</td>
<td>0.010359</td>
<td>0.009983</td>
<td>0.009718</td>
<td>0.009521</td>
<td>0.00937</td>
<td>0.00925</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.016667</td>
<td>0.0105278</td>
<td>0.008556</td>
<td>0.007741</td>
<td>0.007272</td>
<td>0.006968</td>
<td>0.006754</td>
<td>0.006596</td>
<td>0.006474</td>
<td>0.006378</td>
<td></td>
</tr>
<tr>
<td>0.014286</td>
<td>0.0152653</td>
<td>0.0093673</td>
<td>0.007719</td>
<td>0.006954</td>
<td>0.006514</td>
<td>0.00629</td>
<td>0.006029</td>
<td>0.005881</td>
<td>0.005768</td>
<td>0.005678</td>
</tr>
<tr>
<td>0.0125</td>
<td>0.0143613</td>
<td>0.0087012</td>
<td>0.007127</td>
<td>0.006398</td>
<td>0.00598</td>
<td>0.005709</td>
<td>0.005519</td>
<td>0.005379</td>
<td>0.005271</td>
<td>0.005186</td>
</tr>
<tr>
<td>0.011111</td>
<td>0.0136831</td>
<td>0.0082047</td>
<td>0.00687</td>
<td>0.005986</td>
<td>0.005584</td>
<td>0.005324</td>
<td>0.005142</td>
<td>0.004904</td>
<td>0.004822</td>
<td>0.004746</td>
</tr>
<tr>
<td>0.01</td>
<td>0.013156</td>
<td>0.007821</td>
<td>0.006348</td>
<td>0.005669</td>
<td>0.00528</td>
<td>0.005028</td>
<td>0.004852</td>
<td>0.004722</td>
<td>0.004622</td>
<td>0.004543</td>
</tr>
</tbody>
</table>

**ACKNOWLEDGMENT**

VSS Yadavalli is thankful to NRF, South Africa, and, UGC Emeritus fellowship, India respectively, for their financial support.

**REFERENCES**