Green's law and the evolution of solitary waves

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An exact solution to the linear shallow-water wave equation is presented for solitary waves first evolving over constant depth and then over a sloping beach. An asymptotic expansion for the evolution of the maximum amplitude of the wave is shown to reduce to Green's law. This appears to be the first derivation of Green's law specifically for solitary waves and including reflection.

The evolution of long waves on a beach is a classic problem in hydrodynamics. Since the work of Boussinesq[1] there have been numerous attempts at modeling the maximum amplitude evolution until breaking. The existing literature is extensive; the basic phenomenology lends itself to analysis both with asymptotic expansions, with perturbation series, and/or with spectral methods. Many interesting results can be found in Mei,[2] and—more recently—in Freilich and Guza,[3] Liu *et al.*,[4] and Kirby.[5]

Green's law is a classic linear theory result and it describes the evolution of the wave height of periodic waves on plane beaches.[6] If $\eta_{\max}$ is the normalized maximum height of the wave at any x location, Green's law predicts that $\eta_{\max} \sim 1/h^{1/4}$, where h is the local undisturbed water depth, and where $\eta_{\max}$ and h are normalized by some depth d. Green's law is different from the Boussinesq result which predicts that $\eta_{\max} \sim 1/h$, a result that incorporates both dispersion and nonlinearity. The range of validity of the two evolution results is discussed elsewhere.[7] As Miles[8] notes, *the Boussinesq approximation is valid for sufficiently small values of the bottom slope, but Green's law is a better approximation for sufficiently large bottom slope or sufficiently small amplitude.*

The proof of Green's law is a standard exercise in wave theory (Lamb[9]) in the context of the evolution of waves in a channel of a variable cross section. However, it has only been established under steady wave conditions and when, as Lamb notes, "... a wave suffers no appreciable disintegration by reflection... ." In this Brief Communication, an alternate derivation of Green's law will be presented that is valid for solitary waves climbing the topography defined by Fig. 1, i.e., of a constant depth region adjacent to a sloping beach. The solution will include reflection from the abrupt transition in depth.

Consider the linearized form of the one-dimensional shallow-water wave equation

$$\eta_{tt} - (\eta_x h)_x = 0, \tag{1}$$

where $\eta(x,t)$ is the local wave amplitude and $h(x)$ is the local undisturbed water depth. All variables have been normalized by the offshore depth h_0. Keller and Keller[10] derived the steady-state solution to (1) for sinusoidal waves. Synolakis[11,12] argued that since the problem is homogeneous and linear, steady solutions can be superposed to obtain traveling wave solutions. For an incoming wave distribution of the form

$$\eta(x,t) = \int_{-\infty}^{\infty} \Phi(k) e^{-ikct}\, dk, \tag{2}$$

he showed that the amplitude of the wave transmitted to the beach is given by

$$\eta(x,t) = 2 \int_{-\infty}^{\infty} \Phi(k)\, \frac{J_0(2k\sqrt{xX_0})\, e^{-ik(X_0+ct)}}{J_0(2X_0k) - iJ_1(2X_0k)}\, dk, \tag{3}$$

where X_0 is the cotangent of the angle of the beach; in the present normalization, it is also the length between the initial shoreline and the toe of the beach.

Here, an exact representation of the solution of the integral (3) is sought for solitary waves and for large x, i.e., far from the shoreline. Recall that for a solitary wave whose crest is at $x = X_1$ at time $t = 0$, then

$$\Phi(k) = (\tfrac{2}{3}) k \operatorname{cosech}(\alpha k) e^{ikX_1}, \tag{4}$$

where $\alpha = \pi/\sqrt{3H}$.

The solution proceeds using contour integration. The present analysis is a generalization of Synolakis's[11,12] method which only concerned the maximum runup; in that solution $x = 0$.

Consider the integral

$$I(z) = \left(\frac{4}{3}\right) \oint z \operatorname{cosech}(\alpha z) \times \frac{J_0(2z\sqrt{x \cot\beta})\, e^{iz\theta}}{J_0(2z\cot\beta) - iJ_1(2z\cot\beta)}\, dz. \tag{5}$$

The phase θ is given by $\theta = X_1 - \cot\beta - ct$. The contour $\mathscr{C}$ consists of the entire real axis (where $z = x$) and the semicircular contour in the upper half-plane, where $z = Re^{i\phi}$ and $0 < \phi < \pi$. It can be shown by inspection that Jordan's lemma requires that the integral over the semicircular arc go to zero

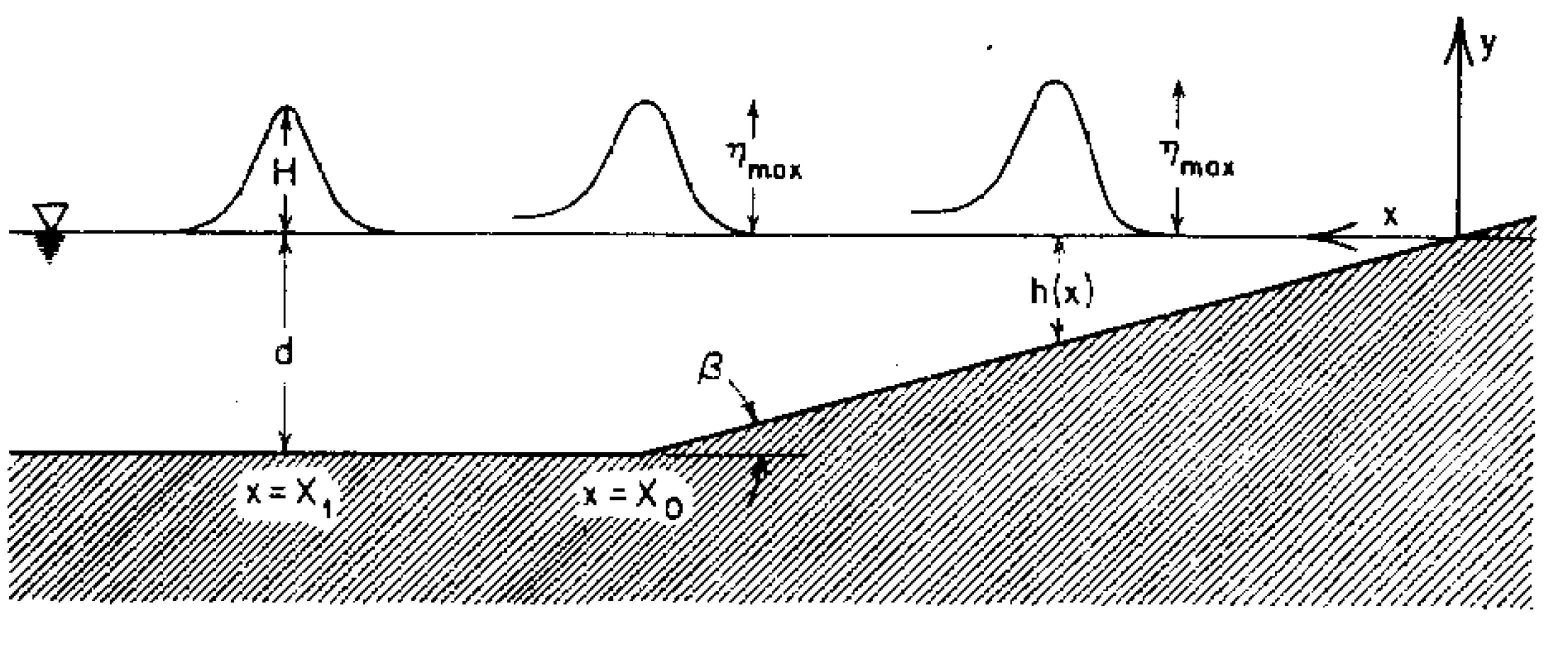

FIG. 1. A definition sketch for the evolution of solitary waves on plane beaches.

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