

# Relational Contract, Limited Liability, and Employment Dynamics\*

Preliminary and Incomplete

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## Abstract

This paper shows that, by adding a limited liability constraint to a standard relational contract model with imperfect public monitoring, one can obtain definitive empirical implications of optimal relational contracts that shed light on important features of employment relationship. First and foremost, the employment dynamics has an “up-or-out” property. An employee begins the employment relationship in a probation phase, and depending on the outputs, either he receives permanent employment or the relationship is terminated forever. Second, the employee’s compensation is deferred in the sense that his expected compensation increases over time even if his expected productivity remain constant. In particular, the bonus payment as a proportion of his total compensation increases with time. Moreover, the turnover rate goes to zero as the employment duration goes to infinity but the turnover rates need not to be monotone. For some cases, the optimal relational contract displays features of tenure: there is a fixed time in which the employee will be terminated if no permanent employment is obtained prior to that. Finally, employment is path dependent. Earlier success is more important both in terms of increasing future expected compensation and in terms of reducing the probability of termination.

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# 1 Introduction

Long-term employments are important for the U.S. economy. Hall (1982) finds that the average job tenure in the U.S. is eight years and a quarter of the workforce hold jobs that last more than twenty years. While workers appear to change jobs more often recently, the importance of long-term employment relationship in the economy has also been reconfirmed by more recent findings; see for example Ureta (1991) and Stevens (2005). Despite the prevalence of long-term employment, long-term contracts appear to be rare. And even for workers with long-term contracts, they are often very incomplete and leave many aspects of the job unspecified. The lack of complete long-term contracts appears puzzling because economic theory suggests that the optimal contract may reward the worker based on all his previous outputs; see for example Rogerson (1985).

One resolution to this apparent paradox is that workers and employers may form relational contracts. Relational contracts are self-enforcing contracts not enforced by the rule of the court but by the concerns of the parties for their future interests. These informal and implicit agreements can powerfully affect the behavior of workers and employers. Bull (1987) shows that relational contract can help sustain an efficient employment relationship even if explicit contracts are absent. In the employment context, theoretical properties of relational contracts have been further explored under different information and contractability assumptions in important papers by Macleod and Malcomson (1989), Baker, Gibbons and Murphy (1994), Levin (2003), and Fuchs (2007).

While the existing models of relational contract have provided important insight into “why relational contracts are important in employment relationships”, they have not yielded definitive empirical implications on “how relational contract affect employment relationship.” (One exception is MacLeod and Malcomson (1998). They show that employers are more likely to choose efficiency wage over performance bonus when it is easier for employers to find new workers) The main reason is that in the existing models, the optimal relational contracts can be implemented in many different ways, so no definitive employment patterns are obtained. For example, Levin (2003) shows that, when outputs are commonly observed, the optimal relational contract can be implemented by stationary contracts, yet many non-stationary contracts are also optimal. When the outputs are observed only by the employer and the employee has private information about his action, Fuchs (2007) shows that the optimal relational contract can be implemented by termination contracts in which no bonuses are ever paid out, yet there are also optimal relational contracts that pay out bonuses.

One reason for the multiplicity of optimal employment patterns is that in these models there are free monetary transfers between employers and employees. While the free transfer assumption facilitates the analysis, it also implies that there are great flexibilities in the ways that

money transfers can be made, and this gives rise to many types of employment patterns. In real employment relationships, however, there are often restrictions on the transfers. Importantly, it is difficult to take money out of an employee’s pocket. Moreover, there are minimum-wage laws that prevent the employers from paying wages below certain thresholds<sup>1</sup>.

This paper shows that, by adding a limited liability constraint to a standard relational contract model with imperfect public monitoring, one can obtain definitive empirical implications of optimal relational contracts that shed light on important features of employment relationship. First and foremost, the employment dynamics has an “up-or-out” property. An employee begins the employment relationship in a probation phase, and depending on the outputs, either he receives permanent employment or the relationship is terminated forever. Second, the employee’s compensation is deferred in the sense that his expected compensation increases over time even if his expected productivity remain constant. In particular, the bonus payment as a proportion of his total compensation increases with time. Moreover, the turnover rate goes to zero as the employment duration goes to infinity but the turnover rates need not to be monotone. For some cases, the optimal relational contract displays features of tenure: there is a fixed time in which the employee will be terminated if no permanent employment is obtained prior to that. Finally, employment is path dependent. Earlier success is more important both in terms of increasing future expected compensation and in terms of reducing the probability of termination.

In this model, there is one principal and one agent. Both are risk neutral and infinitely lived. If they don’t engage in a relationship, both receive their outside options. Otherwise, production takes place. We assume that both the outputs and the efforts are binary, and high outputs are more likely if the agent exerts effort. The effort choice is the agent’s private information. And the output is commonly observed by the principal and the agent, but it is not contractible. So far, this is the standard setup of relational contracts with imperfect public monitoring, as in Levin (2003). The extra restriction we impose is that, there is a lower bound in the transfer the principal can make to the agent, i.e. there is a minimum wage. We solve for the set of Perfect Public Equilibria (PPE) payoffs of this model, and each relational contract corresponds to a PPE.

Under the assumption that a) minimum wage exceeds the agent’s outside option and b) the surplus in the relationship is sufficiently high so efficient relational contract can be supported as a PPE, we completely characterize the Pareto frontier of PPE payoff set. The Pareto frontier can be classified into three regions. To the left of a low threshold, the relationship terminates immediately with positive probability and the Pareto frontier is a straight line with positive slope. To the right of a high threshold, the relational contract attains full Pareto efficiency and

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<sup>1</sup>Several models have analyzed the implications of limited liability in a static setting; see for example Sappington (1983), Innes (1990), Kadan, and Swinkels (2008), and Jewitt, Kadan, and Swinkels (forthcoming).

the Pareto frontier is a straight line with a slope of -1. Between the two thresholds, the Pareto frontier satisfies a functional equation. There is a unique solution to the functional equation, and we show that it can be found using a two-step procedure where each step involves solving a fixed point of a contraction mapping. For some cases, this functional equation can be solved analytically: the Pareto frontier consists of a countable number of line segments. The Pareto frontier is undifferentiable because there is a kink whenever two line segment meets, and this sequence of kinks converge to the right threshold.

The characterization implies that the optimal relational contract, the PPE that maximizes the principal's payoff, is inefficient in the sense that termination occurs with positive probability. This is because when there is limited liability, there is rent in the relationship for the agent. Consequently, the principal can use the threat of termination to induce effort. Since output depends both on effort and luck in this model, termination can occur even if the agent always put in effort. Of course, termination is costly to the principal as well. It is not clear that the principal will prefer using termination over giving bonus to provide incentive at the margin. In fact, when high output is unlikely and we restrict our attention to stationary contracts, the optimal relational contract in this class involves no termination.

The more efficient way to provide incentive is to backload the payoff of the agent. In particular, the optimal relational contract specifies that no bonus will be paid out in the earlier stage of the employment. In this way, even if the relationship is terminated, the principal has captured a bigger share out of surplus in the early stage and termination becomes less costly. Of course, the bonus cannot be postponed forever. At some stage (after some relatively good outcomes), bonus will be paid out and the agent receives permanent employment.

The above is the key intuition of why the structure of employment takes an “up-or-out” form. In addition, it implies why the compensation of the agent is backloaded, and in particular, why bonus as a proportion of total compensation will increase over time. Moreover, since there is randomness in output, this explains there is a rich turnover patterns. But as time goes to infinity, most of the remaining workers have received permanent employment, so the turnover rate goes to zero. Finally, since rewards for good outcomes are postponed in the relationship, earlier success will accrue a bigger interest than later ones, so the employment dynamics is path dependent and earlier success is more valuable.

We also study how the optimal relational contract is affected by the outside options of the players and by the minimum wage. When the agent's outside option improves, the threat of termination becomes less effective in inducing effort. Therefore, the principal will rely more on bonus and terminate less often. This increases the overall efficiency and also the agent's payoff, but it hurts the principal's payoff. Similarly, when the principal's outside option improves,

termination imposes less cost to the principal and is thus used more often, so the overall efficiency decreases, the agent is worse off, but the principal is better off.

When the minimum wage increases, the agent receives more rent in the relationship so termination becomes more effective both because the agent values the relationship more and because the principal suffers a smaller loss upon termination. Consequently, termination becomes more often, the relationship becomes less efficient, and the principal is worse-off. However, the effect of minimum wage on the agent is ambiguous: while the agent is terminated more often, minimum wage increases his rent in the relationship.

While the intuition for the results above is straightforward, the technique for deriving the comparative statics is somewhat non-standard. This is because the Pareto frontier is characterized by a functional equation, so even if the structure of the optimal relational contract is known, the exact details of the contract, and thus the equilibrium payoffs of the principal and the agent are not known except for some special cases. To derive the comparative statics, we develop a technique relying on the properties of functional operators. We believe the technique may be useful elsewhere as well.

As mentioned, the previous results are obtained under the assumption that a) minimum wage exceeds the agent's outside option and b) the surplus in the relationship is sufficiently high so efficient relational contract can be supported as a PPE. When the surplus in the relationship is low, the basic structure of the PPE and the employment dynamics from the optimal relational contract remains very similar. The key difference is that there will not be permanent employment and the relationship will be terminated with probability 1 in finite time. On the other hand, when the minimum wage is lower than the outside option, things are more complicated. For sufficiently low minimum wage, the limited liability has no bite and the optimal relational contract can be implemented by a sequence of stationary contracts just as Levin (2003). For minimum wage a bit lower than the outside option, the structure of the PPE set is preserved and so does the employment dynamics. But for intermediate values of minimum wage, we are still unable to characterize the PPE payoff sets.

This paper contributes to three growing literatures. In the relational contract literature, this paper shows how relational contract can rise and fall under limited liability. The relational contract literature can be broadly classified into three categories: perfect monitoring, in which the effort of the agent can be observed; see for example Macleod and Malcolmson (1989), imperfect public monitoring, in which the effort is private information but outputs are publicly observed; see for example Levin (2003), and private monitoring, in which the outputs are privately observed by the principal; see for example Fuchs (2007). In the first two classes of models, the optimal relational contract never terminates. In other words, relational contracts will arise, but never fall. In the private monitoring model, the relational contract will always fall: Fuchs (2007)

show that the optimal contract can be implemented by a termination contract (and it can be shown that all termination contracts will end in finite time with probability 1). But the analysis is difficult and structure of the optimal contract is unknown. Here, the relationship can rise, can fall, but doesn't have to always fall.

A closely related paper is Thomas and Worrall (2007). They investigate a dynamic relational contract between two partners under limited liability. Their model is more general by having a continuum of effort choices and outputs. The key difference is that the efforts in that model are publicly observed, and consequently, the relationship never ends. Introducing imperfect public monitoring makes it possible to analyze the determinants of the duration of an employment, turnover pattern, the impact of early success on a career path. Moreover, the limited liability assumption in that model is two-sided in the sense that transfers cannot exceed the total output. Here, the principal can (and will in some cases) award the agent with a bonus that exceeds the total output. Finally, that is a model of partners and the moral hazard is two-sided. Here, we study an employment relationship and there is one-sided moral hazard.

This paper also contributes to the literature of formal models that explain multiple patterns of employment patterns in internal labor markets; see for example Harris and Holmstrom (1982); Bernhardt (1995); and Gibbons and Waldman (1998). The existing models typically assume that workers differ by their types, which can be learned by the employed over time. While no doubt that learning is an important factor in determining employment dynamics, we show that important features of employment dynamics, including probation phase, deferred compensation, tenure, and fast track, can also arise out of concerns for dynamic incentives.

Finally, this paper is also related to the literature of dynamic security designs; see for example Biais et al (2007), DeMarzo and Fishman (Forthcoming), and DeMarzo and Sannikov (2006). In that literature, an entrepreneur with limited liability needs to finance a multi-period investment project. Moral hazard arises because the entrepreneur may hide the cash flows from the investors. Using recursive approach, these papers show that the equilibrium Pareto frontier has similar property as that of this paper. Unlike this paper, the analysis of discrete-time models in that setting appears difficult, which is one motivation for the use of continuous-time models in that literature. Here, our model is very tractable: we know how to find the Pareto frontier, how to derive comparative statics, and we have explicit formula of the Pareto frontier for some parameters.

For the rest of the paper, we set up the model in Section 2. We analyze the model in Section 3. The property of the optimal contract is derived in Section 4. Section 5 discusses extensions.

## 2 Setup

Time is discrete and indexed by  $t \in \{1, 2, \dots, \infty\}$ .

### 2.1 Players

There's one principal and one agent. Both are risk neutral, infinitely lived, and have a common discount rate of  $\delta$ . The agent's per period outside option is  $\underline{u}$ ; the principal's per period outside option is  $\underline{v}$ .

### 2.2 Production

If the principal and the agent engages in production together, there is one task that has two outcomes. If the agent puts in effort, the outcome  $Y$  is  $y$  with probability  $p$  and 0 with probability  $1 - p$ . When no effort is put in, the outcome is  $y$  with probability  $q < p$ . The effort costs  $c$ .

We assume that effort is efficient. And moreover, the relationship is less efficient than the outside options if the effort is not put in.

$$py - c > \underline{u} + \underline{v} \geq \underline{v} > qy.$$

### 2.3 Time line and Information

We following the timing as in Levin (2003) with one change that simplifies the exposition. At the beginning of each period  $t$ , the principal decides whether to offer a contract to the agent,  $d_t^P \in \{0, 1\}$ . If the principal chooses not to offer the contract ( $d_t^P = 0$ ), then the two parties receive their outside options.

If the contract is offered, it specifies a wage  $w_t$ . We assume that

$$w_t \geq \underline{w},$$

and this is legally enforceable. There are several ways to interpret this constraint. For example, when  $\underline{w} = 0$ , this constraint can be thought of as a limited liability constraint so that the principal cannot take money out of the agent's pocket. In addition,  $\underline{w}$  can be interpreted as a minimum wage requirement.

The agent chooses  $d_t^A \in \{0, 1\}$ , and if he rejects the contract ( $d_t^A = 0$ ), the two parties receive their outside options. Otherwise, the relationship starts. The agent chooses effort  $e_t \in \{0, 1\}$ , and the output  $Y_t$  is realized.

We assume that the effort is the agent's private information. The output is publicly observed by the principal and the agent, but it cannot be contracted upon.

In some relational contract models, there is a (non-contractible) performance bonus that is paid out at the end of a period<sup>2</sup>. Here, the "performance bonus" is paid at the beginning of next period instead of at the end of the current period. In other words, the agents are incentivized through "efficiency wage" as opposed to "bonus pay". Since all the outputs and bonuses are publicly observed, it is well-known in the literature (see for example Macleod and Malcomson (1988)) that these two setups give rise to the same equilibrium payoff set. While the bonus pay setup helps illustrate the connection between explicit and relational contracts (see for example Levin (2003)), we choose this setup for expositional simplicity.

Finally, we assume that there is a public randomization device, so at the end of each period  $t$ , a random variable  $x_t \in [0, 1]$  will be drawn. This is a standard assumption in the literature made to ensure that the set of perfect public equilibrium payoffs are convex; see for example Fuches (2007).

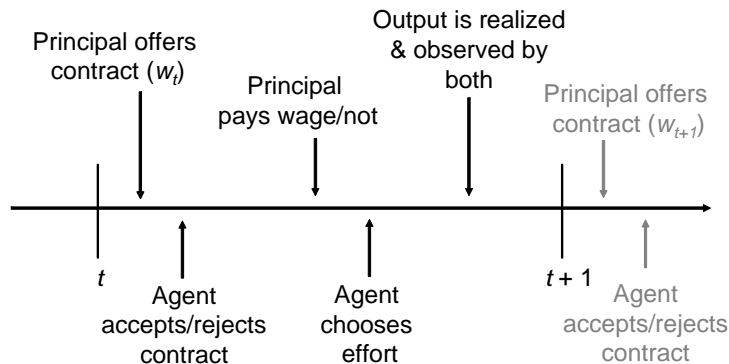


Figure 1: Timeline

## 2.4 Strategy and Equilibrium Concept

### 2.4.1 History

We denote  $h_t = \{d_t^P, w_t, d_t^A, y_t, x_t\}$  as public events that happens in period  $t$ . Denote  $h^t = \{h_n\}_{n=0}^{t-1}$  as a public history path at the beginning of period  $t$ .  $h^1 = \Phi$ . Let  $H^t = \{h^t\}$  be the set of public history paths till time  $t$ . Finally, define  $H = \cup_t H^t$  as the set of public histories.

<sup>2</sup>It is possible that the bonus is negative in those models, and in such case, the agent has a discretion to make payments to the principal.

### 2.4.2 Strategy

We restrict ourselves to public strategies. This means that the actions of a player will be contingent only on events that are publicly observable. In particular, in period  $t$ , the principal chooses  $D_t^P$  from  $H^t$  to  $\{0, 1\}$ ;  $W_t$  from  $H^t$  to  $[\underline{w}, \infty)$ . The public strategy of the principal is  $\{D_t^P, W_t\}_{t=1}^\infty$ .

In period  $t$ , the agent chooses  $D_t^A$  from  $H^t \cup \{w_t\}$  to  $\{0, 1\}$  and  $E_t$  from  $H^t \cup \{w_t\}$  to  $\{0, 1\}$ . And the strategy of the agent is  $\{D_t^A, E_t\}_{t=1}^\infty$ .

We allow the players to play mixed strategy in this game. Denote  $\sigma^A$  to be the mixed public strategy of the agent and  $\sigma^P$  to be the mixed public strategy of the principal. They can be defined in the standard way, so we omit their precise definition here.

### 2.4.3 Public Perfect Equilibrium

We analyze the public perfect equilibrium (PPE) of the game. PPE is the standard equilibrium concept in repeated games of imperfect public monitoring. Moreover, this model is a game of imperfect monitoring with "product structure", in the sense that the output depends on the agent's alone. In this case, our restriction to study only PPEs is without loss of generality; see Fudenberg and Levine (1994).

In particular, a strategy profile is a PPE if a) the players use public strategies, and b) following every public history, the restriction of the strategy is a Nash equilibrium. This implies that to check whether a strategy profile is PPE, it suffices to check one-stage deviation. In particular, a mixed strategy profile  $\sigma$  is a PPE if and only if  $d_t^P, w_t, d_t^A, y_t, x_t$

- Following any history  $h^t, (d_t^P, w_t)$  in the support of  $\sigma^P$  is the best response for the principal, holding all of the rest of the strategy fixed.
- Following any history  $h^t \cup \{d_t^P, w_t\}, (d_t^A, e_t)$  in the support of  $\sigma^A$  is the best response for the agent, holding all of the rest of the strategy fixed.

## 3 Analysis

We solve the model in this section. In Subsection 3.1, we review the basics of a powerful technique developed by Abreu, Pierce, Stacchetti (1990) (APS hereafter) that can be used to characterize the PPE payoff set. And we show that the PPE payoff set is completely determined by its Pareto frontier. In Subsection 3.2, we characterize the Pareto frontier. In Subsection 3.3, we show that the optimal relational contract is inefficient. In Subsection 3.4, we explicitly calculate the Pareto frontier of the PPE set for a range of parameters.

### 3.1 General Method of Finding PPEs and its Simplification

In this subsection, we review the basics of a powerful technique developed by Abreu, Pierce, Stacchetti (1990) (APS hereafter) that can be used to characterize the PPE set. We also show that the standard APS technique in the current setting, which involves looking for a two-dimensional graph, can be simplified into looking for a one-dimensional curve.

Abreu, Pierce, Stacchetti (1990) (APS hereafter) develops a powerful technique in characterizing the PPE payoff set in repeated games with imperfect public monitoring. The basic idea is that a player's payoff in a repeated game can be decomposed into a current payoff and a future continuation payoff. This transforms an infinitely-repeated game into a two stage game. The action of a player maximizes the weighted average of these two. Therefore, knowledge of how the current and continuation payoffs are affected by the actions of the players help characterize the actions of the players in a PPE and thus the PPE payoff set.

In particular, let  $Y$  be the set of publicly observed outcomes in the stage game of a  $n$ -player infinitely repeated game. Let  $W$  be a set of continuation payoff vectors. An action profile  $a^*$  is enforceable with respect to  $W$  if and only if there exists a function  $\gamma : Y \rightarrow W$  such that for all  $i$ ,

$$a_i^* \in \arg \max_{a_i} \int [(1 - \delta)u_i(a_i, a_{-i}^*, y) + \delta\gamma_i(y)]d\mu(y|(a_i, a_{-i}^*)),$$

where  $u_i$  is stage game payoff function for player  $i$ ,  $\gamma_i$  is his continuation payoff, and  $\mu$  is the probability measure of  $y$  induced by the actions.

Similarly, we define a payoff vector  $v$  to be enforceable with respect to  $W$  if there exists  $a^*$  (and an associated  $\gamma$ ) enforceable with respect to  $W$  such that for all  $i$ ,

$$v_i = \int [(1 - \delta)u_i(a^*, y) + \delta\gamma_i(y)]d\mu(y|(a^*)).$$

And we denote

$$B(W) = \{v : v \text{ enforceable with respect to } W\}.$$

The two-stage decomposition of the infinitely repeated game implies that we can think of  $B(W)$  as the set of equilibrium payoffs given that  $W$  is the set of continuation equilibrium payoffs.

Note that  $B$  is a monotone operator in the sense that

$$B(W_1) \subset B(W_2) \text{ if } W_1 \subset W_2.$$

Moreover, APS note that the  $B$  operator is closely linked with the PPE payoff sets. Denote  $E$  as the set of PPE payoffs. Then

$$B(E) = E.$$

To see this, first, it is clear that  $B(E) \subset E$ . This is because every element in  $B(E)$  satisfies that i) it has an equilibrium continuation payoff and ii) the players do not want to deviate in period 1, so it is again an equilibrium payoff. Second,  $E \subset B(E)$  follows because the continuation payoff of each equilibrium is again an equilibrium, and this implies that  $E = B(E')$  for some  $E' \subset E$ . Then the monotonicity of  $B$  implies that  $E = B(E') \subset B(E)$ .

APS also show that  $E$  is convex and compact. Moreover,  $E$  can be found numerically by noting that

$$E = \lim_{n \rightarrow \infty} B^n(F), \text{ where } F \text{ is the set of feasible payoffs.}$$

Note that  $F$  is an  $n$ -dimensional set, so computing  $B(F)$  can be computationally involved. Moreover, the speed of convergence of  $B^n(F)$  is not known, and this adds to the computational complexity. Therefore, it can be computationally challenging to characterize the PPE payoff set.

In this model, two features of the relational contract setup help simplify the characterization of the PPE payoff set. First, both the principal and the agent can choose to take their outside option, so any PPE payoff must give the principal at least  $\underline{v}$  and the agent at least  $\underline{u}$ . Moreover, taking their outside options  $(\underline{u}, \underline{v})$  is a PPE payoff, supported by the belief that the principal will not offer contract to the agent in the future and the agent will never put in effort.

Second, the principal can make transfers to the agent. This implies that, if  $(u, v)$  is a PPE payoff, then extra transfer from the principal to the agent at the beginning of period 1 can produce another PPE payoff  $(u', v')$  with  $u' + v' = u + v$ ,  $u' > u$ , and  $v' \geq v$ . In particular,  $(u + v - \underline{v}, \underline{v})$  is a PPE payoff.

These two features imply that any payoff that a) gives the agent at least  $\underline{u}$ , b) gives the principal at least  $\underline{v}$ , and c) lies below the Pareto frontier of the PPE payoff set, is again a PPE payoff. In other words, the PPE payoff set is completely characterized by its Pareto frontier.

We denote  $f(u)$  as the maximum PPE payoff of the principal if the agent's payoff is  $u$ . From APS, we know that  $f$  is well-defined because the PPE set is compact. The lemma below states the result above formally.

**Lemma 0:** Let  $u_{\max}$  be the maximum PPE payoff of the agent. Then the PPE payoff set  $E$  is given by

$$E = \{(u, v) : u \in [\underline{u}, u_{\max}], v \in [\underline{v}, f(u)]\}$$

**Proof.** First, note that the payoff pair  $(\underline{u}, \underline{v})$  (meaning that the agent's normalized expected payoff is  $\underline{u}$  and the principal's normalized payoff is  $\underline{v}$ ) is in the PPE payoff set. This payoff is

supported by an equilibrium in which on the equilibrium path, the principal and the agent does not start a relationship, and off the equilibrium path, the agent never puts in effort and both the principal and the agent do the start the relationship in the future.

Second, it follows by convexity of the PPE payoff set that any payoff on the line segment between  $(\underline{u}, \underline{v})$  and  $(\underline{u}, f(\underline{u}))$  can be supported as a PPE payoff, and this is the left boundary of the PPE payoff set. Third, because there is no limit in the amount of transfer the principal can make to the agent at the beginning of period 1, it is easily seen that the lower boundary of the PPE set is given by the horizontal line at  $\underline{v}$ . Finally, convexity implies that any equilibrium payoff between  $(u, \underline{v})$  and  $(u, f(u))$  can be obtained. ■

### 3.2 Characterizing the Pareto Frontier

In this subsection, we characterize the Pareto frontier  $f$ . Under the assumption that a) minimum wage exceeds the agent's outside option and b) the surplus in the relationship is sufficiently high so efficient relational contract can be supported as a PPE, we show that the Pareto frontier can be classified into three regions. To the left of a low threshold, the relationship terminates immediately with positive probability, and the Pareto frontier is a straight line with a positive slope that starts out at  $(\underline{u}, \underline{v})$ . To the right of a high threshold, the Pareto frontier is a straight line with a slope of -1. In this region, the relational contract attains the unconstrained Pareto efficiency and termination never occurs. Between the two thresholds, the Pareto frontier satisfies a functional equation. There is a unique solution to the functional equation, and we show that it can be found using a two-step procedure where each step involves solving a fixed point of a contraction mapping.

In Subsection 3.2.1, we list the incentive constraints for the principal and the agent. In Subsection 3.2.2, we characterize the Pareto frontier of the PPE payoff with free transfer case. Based on this characterization, we show that Pareto frontier of the PPE payoff with limited liability must satisfy a functional equation. In Subsection 3.2.3, we show that there is a unique solution to the functional equation, and we show that it can be found using a two-step procedure where each step involves solving a fixed point of a contraction mapping. All the proofs are collected in the Appendix.

To simplify our analysis, we are going to first make two assumptions. We discuss in Section 5 how results change when these two assumptions aren't satisfied.

**Assumption 1:**

$$py - \frac{[1 - \delta(1 - p)]c}{\delta(p - q)} \geq \underline{v} + \underline{w}.$$

This assumption insure that there is sufficient surplus in the relationship, so the efficient outcome can be sustained as an equilibrium outcome. As will be seen in Section 5, the basic structure

of the PPE payoff set is preserved even if this assumption fails, but some of the qualitative predictions will differ.

**Assumption 2:**

$$\underline{w} \geq \underline{u}.$$

This assumption states that the outside option of the agent is sufficiently low so that the limited liability has a bite. We make this assumption mainly for analytical convenience, but it is a realistic assumption to make in many real life economic situations, including the case in which  $\underline{w}$  is interpreted as a minimum wage.

**3.2.1 Incentive Constraints**

In the efficient outcome of the game, effort is put in each period. For the agent to have incentive to put in effort in any period, we must have

$$(1 - \delta)(w - c) + \delta((1 - p)u_L + pu_H) \geq (1 - \delta)w + \delta((1 - q)u_L + qu_H),$$

where  $u_H$  corresponds to the agent's continuation payoff after a good outcome and  $u_L$  corresponds to the agent's continuation payoff after a bad outcome. We can rewrite expression above can be rewritten as

$$u_H - u_L \geq \frac{(1 - \delta)c}{\delta(p - q)} \equiv k.$$

Here, the  $k$  is defined as the "bonus" necessary for the agent to put in effort. In other words, to induce the agent to put in effort, the difference in continuation payoffs should differ by at least  $k$ .

In this model, the principal's IC for willing to stay in the relationship at the beginning of period  $t$  is simply

$$v_t \geq \underline{v},$$

where  $v_t$  is the principal's expected payoff at the beginning of period  $t$  (following some public history  $h^t$ ).

Note that in models where the bonus is paid out at the end of a period, the IC constraint of the principal (often called the non-reneging constraint) is given by

$$(1 - \delta)(-b_{t-1}) + \delta v'_t \geq \delta \underline{v},$$

where  $b_{t-1}$  is the bonus paid to the agent and  $v'_t$  is the principal's payoff at the beginning of

period  $t$  after the bonus is paid out. In our model, we have

$$v_t = v'_t - \frac{(1 - \delta)}{\delta} b_{t-1},$$

accounting for the fact that the bonus is paid out at the beginning of period  $t$ .

### 3.2.2 Comparing PPE w/ and w/o Limited Liability

In this subsection, we investigate the shape of the Pareto frontier with limited liability. First, we show that the Pareto frontier of the relational contract with free transfers (Levin (2003)) is a negative 45 degree line. Then we show that the limited liability constraint truncates the negative 45 degree line and turn it into three regions.

When the transfer is free, we can use the transfer at the beginning of the first period to reallocate the payoffs between the principal and agent, as long as their payoffs are greater than or equal to their outside options. More formally, if  $(u, v)$  is a PPE payoff, then all the payoffs in the set

$$\{(u', v') | u' + v' = u + v, u' \geq \underline{u}, v' \geq \underline{v}\}$$

are again PPE payoffs.

This implies that every PPE payoff generates a  $-45$  degree line segment such that all points on the line segment are again PPE payoffs. In particular, if we take the most efficient PPE payoff<sup>3</sup>, then every point on the  $-45$  degree line segment generated by the most efficient PPE are again PPE payoffs. Moreover, these points are the Pareto frontier because they are the most efficient PPE payoffs by definition and any payoff to the upper-right of the line segment gives higher payoff and thus cannot be a PPE payoff.

Assumption 2 implies that the efficient outcome can be sustained as a PPE. Consequently, the Pareto frontier in the case of free transfer is given by

$$\{(u, v) | u + v = py - c, u \geq \underline{u}, v \geq \underline{v}\}$$

It follows that the PPE payoff vector that gives the principal the highest payoff is  $(\underline{u}, py - c)$ . Levin (2003) shows that the optimal relational contract can be sustained by stationary contracts. In this case, payoff vector can be supported by the following. The principal pays the agent a base wage  $w_s$ , and in case of a good outcome, a bonus of  $k$  will be paid out at the beginning of next period. In other words, the next period wage of the agent is  $w_s$  in case of a bad outcome

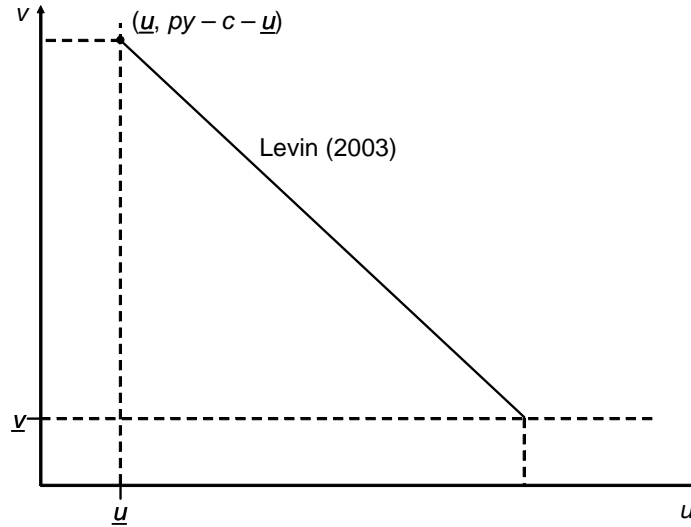
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<sup>3</sup>There can many payoffs that are most efficient. The compactness of the PPE payoff set gurantees that we have at least one.

and  $w_s + k$  in case of a good outcome. The bonus induces the agent to put in effort in each period. And when the base wage

$$w_s = \underline{u} + c - \frac{\delta pk}{1 - \delta} = \underline{u} - \frac{q}{p - q}c.$$

It can be checked that the agent's expected payoff is  $\underline{u}$ , and this leads to a PPE payoff of  $(\underline{u}, py - c - \underline{u})$ .



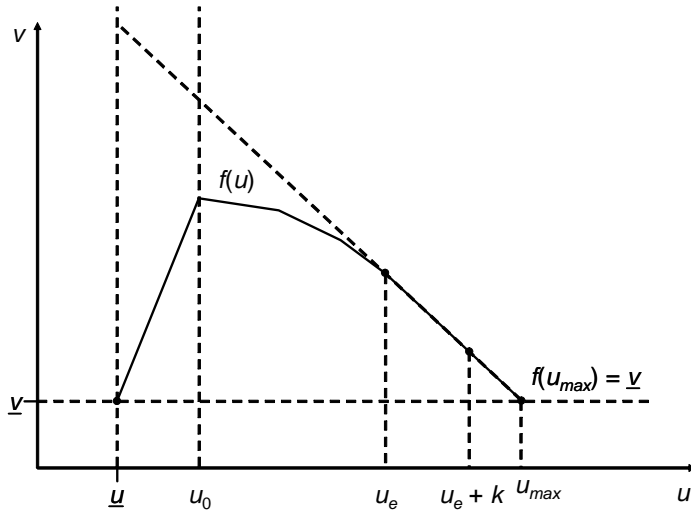
**Figure 2: Pareto Frontier, w/o Limited Liability**

Note that

$$w_s < \underline{u} \leq \underline{w},$$

so the above PPE with free transfer violates the limited liability constraint.

When there is a limited liability constraint, the Pareto frontier is no longer a  $-45$  degree line. Instead, it has three regions.



**Figure 3: Possible Pareto Frontier, w/ Limited Liability**

First, to the right of a threshold ( $u_e$ ) the Pareto frontier is again a  $-45$  degree line. Define  $u_e$  as the unique payoff of the agent such that

$$u_e = (1 - \delta)(\underline{w} - c) + \delta(u_e + pk). \quad (1)$$

In particular,  $u_e$  is obtained when the agent is paid a base wage  $\underline{w}$ , puts in effort, and receives a bonus  $k$  (to be paid out at the beginning of the next period) if the outcome is good. It follows that  $(u_e, py - c - u_e)$  is a PPE payoff and it is efficient. Moreover, first period transfer *from the principal to the agent* insures that the Pareto frontier is a  $-45$  degree line and is efficient for  $u > u_e$ . This implies, in particular, that the relationship will not be terminated for  $u > u_e$ .

At  $u_e$ , the base wage of the agent is  $\underline{w}$ , so the limited liability constraint binds. Therefore, for  $u < u_e$ , the Pareto frontier can no longer lie on the  $-45$  degree line because this would require the agent receive wage lower than  $\underline{w}$  and thus violates the limited liability constraint.

To the left of  $u_e$ , the Pareto frontier can be classified into two regions. In particular, define  $u_0$  such that

$$u_0 = (1 - \delta)(\underline{w} - c) + \delta(\underline{u} + pk). \quad (2)$$

The expression states that if the agent receives a base of  $\underline{w}$ , puts in effort, and receives a continuation payoff of  $\underline{u}$  for a bad outcome and  $\underline{u} + k$  for a good outcome, then the agent's current expected payoff is  $u_0$ . Since the agent cannot receive a payoff less than  $\underline{u}$ , this implies that  $u_0$  is the lowest expected payoff required for the agent to exert effort.

Therefore, for  $u < u_0$ , pure strategy cannot induce the agent to exert effort. It follows that the Pareto frontier for  $u < u_0$  is obtained by a randomization between  $(\underline{u}, \underline{v})$  and  $(u_0, f(u_0))$ ,

where recall that  $f(u_0)$  is the highest PPE payoff for the principal when the agent receives  $u_0$ . The randomization implies that the Pareto frontier is a straight line. In other words, for  $u < u_e$ , there is some positive probability that the relationship will be terminated. And the termination probability corresponds to the weight of  $(\underline{u}, \underline{v})$  in the randomization.

For  $u \in [u_0, u_e]$ , we note that at the Pareto frontier  $((u, f(u)))$ , the wage of the agent in the first period must equal to  $\underline{w}$ . Otherwise, the principal can lower his first period transfer by some small amount, so the slope of the Pareto frontier ( $f$ ) at  $u$  is at most  $-1$ . This cannot be possible because a) the PPE payoff set is convex so  $f$  is concave and b) the slope of  $f$  is strictly less than  $-1$  for  $u < u_e$ .

Since the agent is paid wage  $\underline{w}$  in period 1, we introduce the following linear operator. Define  $L$  as the unique linear operator such that

$$u = (1 - \delta)(\underline{w} - c) + \delta[(1 - p)L(u) + p(L(u) + k)]. \quad (3)$$

In other words,  $L(u)$  corresponds to the agent's continuation payoff next period if he is paid  $\underline{w}$  this period, puts in effort, but the outcome is  $Y = 0$ .

Now if  $((u, f(u)))$  is on the Pareto frontier, this implies that the continuation payoffs must lie on the Pareto frontier as well. In other words, the continuation payoffs corresponding to the good and bad outcomes must be  $(L(u) + k, f(L(u) + k))$  and  $(L(u), f(L(u)))$ . This gives rise to the following functional equation that the Pareto frontier must satisfy:

$$f(u) = (1 - \delta)(py - \underline{w}) + \delta[pf(L(u) + k) + (1 - p)f(L(u))]. \quad (4)$$

Note that for  $u \in [u_0, u_e]$ , the probability that the relationship immediately terminated is zero, but there will be some probability that the relational contract will be terminated eventually.

To summarize, we see that the Pareto frontier of the PPE payoff satisfies

$$f(u) = \begin{cases} \underline{v} + \frac{u - \underline{u}}{u_0 - \underline{u}}(f(u_0) - \underline{v}) & \text{if } u \in [\underline{u}, u_0] \\ (1 - \delta)(py - \underline{w}) + \delta[pf(L(u) + k) + (1 - p)f(L(u))] & \text{if } u \in [u_0, u_e] \\ f(u_e) + u_e - u & \text{if } u \in [u_e, u_e + f(u_e) - \underline{v}], \end{cases}$$

where  $u_e = (\underline{w} - c) + \frac{\delta pk}{(1 - \delta)}$ ,  $f(u_e) = py - c - ((\underline{w} - c) + \frac{\delta pk}{(1 - \delta)})$ , and  $u_0 = (1 - \delta)(\underline{w} - c) + \delta(\underline{u} + pk)$ .

In the next section, we show that there is a unique solution that satisfies the functional equation and how we can find the solution. In Subsection 3.3, we constructed an explicit solution for some cases. Readers who are more interested in the applications of the model can directly jumped to Section 4.

### 3.2.3 Algorithm for Finding the Pareto Frontier

In this subsection, we describe how one can find an  $f$  that satisfies (4). The procedure takes two steps.

In the first step, define an operator  $T_Z$  on the space of bounded functions on  $[u_0, u_e]$  as:

$$T_Z g(u) = \begin{cases} (1 - \delta)(py - \underline{w}) + \delta(pg(L(u) + k) + (1 - p)g(L(u))) & u_0 \leq L(u) < u_e - k \\ (1 - \delta)(py - \underline{w}) + \delta(pg(L(u) + k) + (1 - p)(\underline{v} + \frac{L(u) - \underline{u}}{u_0 - \underline{u}}(Z - \underline{v}))) & L(u) < \min\{u_e - k, u_0\} \\ (1 - \delta)(py - \underline{w}) + \delta(pf(L(u) + k) + (1 - p)g(L(u))) & L(u) > \max\{u_e - k, u_0\} \\ (1 - \delta)(py - \underline{w}) + \delta(pf(L(u) + k) + (1 - p)(\underline{v} + \frac{L(u) - \underline{u}}{u_0 - \underline{u}}(Z - \underline{v}))) & u_0 > L(u) \geq u_e - k \end{cases}, \quad (5)$$

where note that when  $L(u) + k \geq u_e$ ,  $f(L(u) + k) = py - c - (L(u) + k)$ .

Note that if  $Z = f(u_0)$ , then  $f$  restricted to  $[u_0, u_e]$  is a fixed point of  $T_Z$  :

$$\begin{aligned} T_Z f(u) &= (1 - \delta)(py - \underline{w}) + \delta(pf(L(u) + k) + (1 - p)f(L(u))) \\ &= f(u). \end{aligned}$$

But we can also define  $T_Z$  for arbitrary  $Z$ . It can be checked that  $T_Z$  is a contraction mapping in the sense that, if we take any two bounded functions  $g_1$  and  $g_2$  on  $[u_0, u_e]$ , we have

$$\|T_Z g_1 - T_Z g_2\| \leq \delta \|g_1 - g_2\|,$$

where  $\|g\| = \sup_{u \in [u_0, u_e]} |g(u)|$ . Standard fixed point theorem imply that for each  $T_Z$ , there exists a unique  $g_Z$  such that

$$T_Z g_Z = g_Z.$$

In other words, take any  $Z$ , we can define  $T_Z$  and find  $g_Z$ .

The second step involves how to find  $Z^*$  such that  $Z^* = f(u_0)$ . Now for each  $Z$ , the first step induces a fixed point  $g_Z$ . And in particular, this gives a value of  $g_Z(u_0)$ . In the appendix, we show that

$$0 \leq \frac{dg_Z(u_0)}{dZ} < 1.$$

In other words, the function from  $Z$  to  $g_Z(u_0)$  is again a contraction mapping. And it has a unique solution such that

$$Z^* = g_{Z^*}(u_0) = f(u_0) = g_{f(u_0)}(u_0). \quad (6)$$

And in particular, the contraction mapping above is montone, so  $Z^*$  can be found using standard numerical procedures.

In summary,  $f(u_0)$  can be found as follows. First, for each  $Z$ , we map  $Z$  into  $g_Z(u_0)$  through finding  $g_Z$  as a fixed point of the contraction mapping  $T_z$  (defined in (5)). Second, we find  $f(u_0)$  by noting that it is the fixed point of the contraction mapping from  $Z$  into  $g_Z(u_0)$ . We call this procedure "finding a double fixed point".

**Theorem 1:** *The Pareto frontier of the PPE payoff satisfies*

$$f(u) = \begin{cases} \underline{v} + \frac{u-\underline{u}}{u_0-\underline{u}}(f(u_0) - \underline{v}) & \text{if } u \in [\underline{u}, u_0] \\ T_{Z^*}f(u) & \text{if } u \in [u_0, u_e] \\ f(u_e) + u_e - u & \text{if } u \in [u_e, u_e + f(u_e) - \underline{v}], \end{cases}$$

where  $u_e = (\underline{w} - c) + \frac{\delta pk}{(1-\delta)}$ ,  $f(u_e) = py - c - ((\underline{w} - c) + \frac{\delta pk}{(1-\delta)})$ , and  $u_0 = (1-\delta)(\underline{w} - c) + \delta(\underline{u} + pk)$ . Finally,  $f(u_0) = Z^*$ , where  $Z^*$  satisfies  $Z^* = g_{Z^*}(u_0)$  and  $g_Z$  is the fixed point of the operator defined in (5). Moreover, the equation above has a unique solution.

### 3.2.4 Inefficiency in Optimal Relational Contract

In this subsection, we show that the optimal relational contract under limited liability is inefficient. We define the optimal relational contract as the PPE that maximizes the principal's payoff. In addition to serving a theoretical purpose, the inefficiency result will also be used to characterize the employment dynamics in the next section.

**Proposition 1:** *There exists  $u < u_e$  such that*

$$f(u) > f(u_e).$$

**Proof.** Take  $u_0 < u_e$  such that  $f(u_0)$  is differentiable and  $L(u_0) + k > u_e$ . (Note this is a different  $u_0$  from the previous section.) If  $f'(u_0) < 0$ , then we are done, because

$$f(u_0) - f(u_e) = - \int_{u_0}^{u_e} f'(u) du,$$

where this is a Lebesgue integral, and we have  $f'(u) < 0$  due to the concavity of  $f$ .

Otherwise, we know that  $f'(u_0) < \infty$ . Take  $u_1'$  such that  $L(u_1) = u_0$ . According to (7) and the fact that  $u_0 < u_e < u_0 + k$ , within the interval of  $[u_1, u_1 + u_e)$ , we can find  $u_1$  at which

$$f'(u_1) \leq -p + (1-p)f'(u_0).$$

This procedure can continue forever, i.e., we can find  $u_{n+1} \in (u_n, u_e)$  such that

$$f'(u_{n+1}) \leq -p + (1-p)f'(u_n)$$

It is then clear that there exists an  $N$  such that  $f'(u_N) < 0$ , and we are done. ■

In Theorem 1, we know that the right derivative of  $f(u_e)$  is  $-1$ . Proposition 1 essentially shows that the left derivative of  $f(u_e)$  is also  $-1$ , even if  $f$  has infinitely many kinks in any open neighborhood of  $u_e$ . Since  $u_e$  is defined as the smallest payoff of the agent that maximizes the joint surplus, Proposition 1 implies that the optimal contract cannot be efficient.

The characterization implies that the optimal relational contract, the PPE that maximizes the principal's payoff, is inefficient in the sense that termination occurs with positive probability. This is because when there is limited liability, there is rent in the relationship for the agent. Consequently, the principal can use the threat of termination to induce effort. Since output depends both on effort and luck in this model, termination can occur even if the agent always put in effort. Of course, termination is costly to the principal as well. It is not clear that the principal will prefer using termination over giving bonus to provide incentive at the margin. In fact, when high output is unlikely and we restrict our attention to stationary contracts, the optimal relational contract in this class involves no termination.

The more efficient way to provide incentive is to backload the payoff of the agent. In particular, the optimal relational contract specifies that no bonus will be paid out in the earlier stage of the employment. In this way, even if the relationship is terminated, the principal has captured a bigger share out of surplus in the early stage and termination becomes less costly. Of course, the bonus cannot be postponed forever. At some stage (after some relatively good outcomes), bonus will be paid out and the agent receives permanent employment.

### 3.3 Pareto Frontier with Countable Number of Kinks

In Subsection 3.2, we show that  $f$  can be found by solving for a double fixed point problem. In this subsection, we show that, for some parameters, it is possible to calculate  $f$  explicitly.

The condition for the explicit calculation is that

$$\underline{u} + k = L(u_0) + k \geq u_e = (\underline{w} - c) + \frac{\delta pk}{(1 - \delta)}.$$

Or equivalently,

$$\underline{u} + \frac{(1 - \delta - \delta q)}{\delta(p - q)}c \geq \underline{w}.$$

This condition states that the expected payoff of the agent is greater than or equal to  $u_e$  following any good outcomes. And the condition is more likely to be satisfied when the probability of success ( $p$ ) is small and when the discount rate ( $\delta$ ) is small.

First, we state the following corollary.

**Corollary 1:** For almost all  $u \in [u_0, u_e]$ ,

$$f'(u) = pf'(L(u) + k) + (1 - p)f'(L(u)). \quad (7)$$

**Proof.** Because  $f$  is concave, the left derivative is equal to the right derivative almost everywhere. (Royden, Chapter 5). The equation follows directly Lemma 6. ■

In this case, Corollary 1 implies that for  $u \in [u_0, u_e]$ , we have

$$f'(u) = -p + (1 - p)f'(L(u)).$$

From the formula above, we can partition the interval of  $[u_0, u_e]$  into countable number of sub-intervals and then calculate the value of  $f'(u)$  sub-interval by sub-interval.

In particular, define  $u_1$  such that

$$L(u_1) = u_0.$$

Then we have that

$$L(u) \in [\underline{u}, u_0] \text{ for all } u \in [u_0, u_1].$$

Moreover, some algebra gives that

$$u_1 - u_0 = \delta(u_0 - \underline{u}) = \delta((1 - \delta)(\underline{w} - c - \underline{u}) + \delta pk).$$

Next define  $u_{n+1}$  such that

$$L(u_{n+1}) = u_n.$$

Then it can be checked that

$$L(u) \in [u_{n-1}, u_n] \text{ for all } u \in [u_n, u_{n+1}].$$

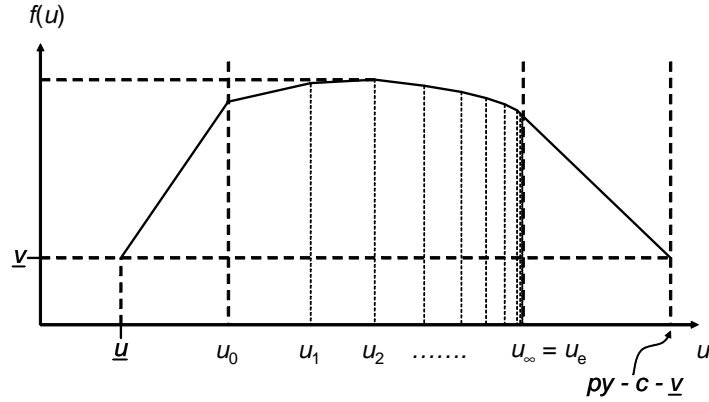
Moreover, some algebra gives that

$$u_{n+1} - u_n = \delta^{n+1}(u_0 - \underline{u}).$$

And it is easy to check that

$$\lim_{n \rightarrow \infty} u_n = u_e.$$

In other words, the  $\{u_n\}_{n=1}^{\infty}$  partitions  $[u_0, u_e]$ . And we are ready to calculate the slope of  $f$  in each sub-interval.



**Figure 4: Pareto Frontier w/ Countable Number of Kinks**

First, define

$$s_0 = \frac{f(u_0) - v}{u_0 - \underline{u}}.$$

Then by the equation on the derivative of  $f$ , we have

$$f'(u) = -p + (1-p)s_0 \quad \text{for } u \in (u_0, u_1).$$

Define  $s_{n+1}$  as the slope of  $f$  in  $(u_n, u_{n+1})$ : then we have

$$\begin{aligned} s_1 &= -p + (1-p)s_0; \\ s_{n+1} &= -p + (1-p)s_n. \end{aligned}$$

Now note that

$$s_{n+1} - s_n = (1-p)(s_{n+1} - s_n),$$

so

$$s_n = s_0 - (1 + s_0)(p + (1 - (1-p)^{n+1})).$$

Now note that for  $u \in [u_n, u_{n+1})$ ,

$$\begin{aligned} f(u) &= f(u_0) + \sum_{j=1}^n (u_j - u_{j-1})s_j + s_{n+1}(u - u_n) \\ &= v + s_0(u_0 - \underline{u}) + \sum_{j=1}^n \delta^j (u_0 - \underline{u})s_j + s_{n+1}(u - u_n) \\ &= v + (u_0 - \underline{u}) \sum_{j=0}^n \delta^j s_j + s_{n+1}(u - u_n). \end{aligned}$$

The summation  $\sum_{j=0}^n \delta^j s_j$  can be evaluated explicitly. Define

$$S(n) = \sum_{j=0}^n \delta^j s_j.$$

Then

$$\begin{aligned} (1 - \delta(1 - p))S(n) &= s_0 + \sum_{j=1}^n \delta^j (s_j - (1 - p)s_{j-1}) - \delta^{n+1} s_n (1 - p) \\ &= s_0 - \frac{p\delta(1 - \delta^{n+1})}{1 - \delta} - \delta^{n+1} s_n (1 - p). \end{aligned}$$

Substituting this back into the expression for  $f(u)$ , we have

$$\begin{aligned} f(u) &= \underline{v} + (u_0 - \underline{u}) \sum_{j=0}^n \delta^j s_j + s_{n+1}(u - u_n). \\ &= \underline{v} + \frac{(u_0 - \underline{u})}{1 - \delta(1 - p)} \left( s_0 - \frac{p\delta(1 - \delta^{n+1})}{1 - \delta} - \delta^{n+1} s_n (1 - p) \right) + s_{n+1}(u - u_n). \end{aligned}$$

Since  $s_n = s_0 - (1 + s_0)(p + (1 - (1 - p)^{n+1}))$ , the expression above determines  $f$  completely as long as we know what  $s_0$  is. Note that

$$\begin{aligned} \underline{v} + s_0(u_0 - \underline{u}) &= f(u_0) \\ &= (1 - \delta)(py - \underline{w}) + \delta((1 - p)\underline{v} + p(py - c - (\underline{u} + k))), \end{aligned}$$

where the second inequality uses Lemma 6. This implies that<sup>4</sup>

$$s_0 = \frac{(1 - \delta)(py - \underline{w}) + \delta((1 - p)\underline{v} + p(py - c - (\underline{u} + k))) - \underline{v}}{(1 - \delta)(\underline{w} - c) + \delta(\underline{u} + pk) - \underline{u}}.$$

And we have the following result.

---

<sup>4</sup>An alternative method of calculating  $s_0$  is to note that

$$f(u_e) = f(u_0) + \sum_{n=1}^{\infty} s_n(u_n - u_{n-1}) = \underline{v} + s_0(u_0 - \underline{u}) + \frac{\delta u_0(-p + (1 - p)s) - \frac{p\delta^2 u_0}{1 - \delta}}{1 - (1 - p)\delta}.$$

And in addition,

$$f(u_e) = py - c - ((\underline{w} - c) + \frac{\delta pk}{(1 - \delta)}).$$

Some algebra shows that this gives the same  $s_0$  as before.

**Corollary 2:** If  $\underline{u} + \frac{(1-\delta-\delta q)}{\delta(p-q)}c \geq \underline{w}$ , then

$$f(u) = \begin{cases} \underline{v} + s_0(u - \underline{u}) & \text{if } u \in [\underline{u}, u_0] \\ \underline{v} + \frac{(u_0 - \underline{u})}{1-\delta(1-p)}(s_0 - \frac{p\delta(1-\delta^{n+1})}{1-\delta} - \delta^{n+1}s_n(1-p)) + s_{n+1}(u - u_n) & \text{if } u \in [u_n, u_{n+1}] \\ f(u_e) + u_e - u & \text{if } u \in [u_e, u_e + f(u_e) - \underline{v}], \end{cases}$$

where  $u_0 = (1-\delta)(\underline{w} - c) + \delta(\underline{u} + pk)$ ,  $s_0 = \frac{(1-\delta)(py - \underline{w}) + \delta((1-p)\underline{v} + p(py - c - (\underline{u} + k)) - \underline{v})}{(1-\delta)(\underline{w} - c) + \delta(\underline{u} + pk) - \underline{u}}$ ,  $u_n = u_0 + \frac{\delta(1-\delta^n)}{1-\delta}(u_0 - \underline{u})$ ,  $s_n = s_0 - (1 + s_0)(p + (1 - (1-p)^{n+1}))$ ,  $u_e = (\underline{w} - c) + \frac{\delta pk}{(1-\delta)}$ ,  $f(u_e) = py - c - ((\underline{w} - c) + \frac{\delta pk}{(1-\delta)})$ .

Corollary 2 implies that the graph of  $f$  has a kink at each of the  $u_n$ . Therefore,  $f$  is not differentiable at each point. Interestingly, contrary to most examples in the literature where the PPE payoff set typically has finite or a continuum of extremal points, we have a countable number of extremal points here, and they converge to  $u_e$ .

## 4 Empirical Implications

In this section, we derive some properties of the optimal relational contract using the characterization of the PPE frontier in the previous section. In Subsection 4.1, we characterize the employment dynamics of the relationship. In Subsection 4.2, we derive comparative static results on how the outside options and the minimum wage affects the principal, agent's payoffs and the overall efficiency. All these implications are derived under Assumption 1 and 2, and we discuss in the next section how the predictions change under more general conditions.

### 4.1 Patterns of Employment Dynamics

In this subsection, we derive the empirical implications of optimal relational contract. In Subsection 4.1.1, we show that the employment dynamics has an “up-or-out” property. In Subsection 4.1.2, we show that the employee's compensation is deferred in the sense that his expected compensation increases over time even if his expected productivity remain constant. In Subsection 4.1.3, we explore the turnover dynamics. Sometimes, the optimal relational contract displays features of tenure: there is a fixed time in which the employee will be terminated if no permanent employment is obtained prior to that. Subsection 4.1.4 shows show that the employment is path dependent.

#### 4.1.1 Probation and Up-or-Out

The next proposition characterizes the employment dynamics of the optimal relational contract. While the optimal relational contract with limited is not stationary, it is still relatively simple.

Essentially, the agent's employment dynamics can be categorized into three phases. The agent starts with Phase 1, in which he is paid the minimum wage  $\underline{w}$  per period. Depending on the random outcomes, the agent either moves into Phase 2, in which the relationship is terminated and the two parties receive outside option  $(\underline{u}, \underline{v})$  forever. Or the agent moves into Phase 3, in which the relationship will never be terminated. In this phase, the remaining relational contract can be implemented stationarily: the agent receives a fixed base wage, and he receives a bonus (to be paid out at the beginning of next period in our setting) if the output is high. Moreover, Phase 2 and Phase 3 are absorbing in the sense that, as the time goes to infinity, the agent will either be in one of the two phases with probability 1, and once the agent is in that phase, he stays there forever.

**Proposition 2:** *In the optimal relational contract, the set of histories can be partitioned into  $H = H_1 \cup H_2 \cup H_3$ , such that*

(i): *When  $h^t \in H_1$ ,  $w(h^t) = \underline{w}$ .*

(ii): *When  $h^t \in H_2$ , both the principal and the agent receive their outside options  $(\underline{u}, \underline{v})$ .*

(iii): *When  $h^t \in H_3$ , the optimal contract can be implemented in the following way:*

$$\begin{aligned} w(h^{t+1}) &= \underline{w} && \text{if } h_t = 0; \\ &= \underline{w} + k && \text{if } h_t = y. \end{aligned}$$

(iv): *If  $h^t \in H_i$ , for  $i = 2, 3$ , then  $h^{t+k} \in H_i$  if  $h^{t+k}|_t = h^t$ .*

(v):

$$h^1 \in H_1.$$

(vi):

$$\lim_{t \rightarrow \infty} \Pr(h^t \in H_2 \cup H_3) = 1.$$

**Proof.** Define  $H_3$  as the set of histories such that the agent's continuation payoff  $u \geq u_e$ . Define  $H_2$  as the set of histories such that the agent's continuation payoff  $u = \underline{u}$ . By Theorem 1, it is clear  $H_2 \cap H_3 = \Phi$ . Define  $H_1 = H \setminus (H_2 \cup H_3)$ .

By Proposition 1, we know that the game starts in  $H_1$ , so (v) is proved. Theorem 1 also directly gives (i). Since  $(\underline{u}, \underline{v})$  is the unique PPE payoff in which the agent's payoff is  $\underline{u}$  and it is supported by taking outside option forever, (ii) is proved. Now if  $u \geq u_e$ , then  $f(u) + u = py - c$ , so the continuation payoff must be  $py - c$  as well. Moreover,  $(u, f(u))$  can be implemented by paying the agent  $\underline{w} + \frac{u - u_e}{1 - \delta}$ , in this period, and uses  $(u_e, f(u_e))$  and  $(u_e + k, f(u_e) - k)$  as continuation payoffs forever. This proves (iii) and (iv). (vi) follows because take any  $u$ , there exists an  $N$  such that with a fixed probability bounded away from 0, the agent either have continuation payoff  $(u, v)$  or  $u \geq u_e$ . Then (vi) follows from standard statistics arguments. ■

The three phases in the employment dynamics correspond to the three regions in the PPE

payoff set. The PPE starts in the middle region, and the continuation payoffs bounce around according to the outcomes. If the outcome is good, the continuation payoff moves to the right. Otherwise, it moves to the left. When the continuation payoff moves across the right threshold ( $u_e$ ) the agent receives permanent employment, and the remaining optimal contract can be implemented in a sequence of stationary contracts (cf Levin (2003)). When the continuation payoff cross the left threshold ( $u_0$ ), then termination occurs with some probability.

Phase 1 of the employment resembles the probation period in employment contracts, which are often defined as the period during which the employee can be fired without cause. The probation phase is an important feature for many occupations, including lawyers, doctors, professors, and government officials. The probation periods have received some attention from labor economists; including Bull and Tedeschi (1989), Sadanand et al. (1989), Weiss and Wang (1990), and Wang and Weiss (1998). Most of these models assume that workers differ by their types and the probation period is used as a sorting device to screen out the bad types. And no doubt that this is an important function of the probation period; see for example Loh (1994).

Our analysis suggests that the probation period can also serve as an incentive device. Workers exert effort in the probation phase in hope of receiving permanent employment as reward. Riphahn and Thalmaier (1999) find significant responses of white collar employees and public sector workers to probation periods: Once employment probation is completed and individuals enter into regular employment contracts, the probability of work absences takes discrete jumps and is significantly above previous levels.

In addition, the previous theoretical literature implies that there is a fixed duration for the probation period, and our analysis suggests that the probation period can be random. Casual empiricism suggests that while the probation periods in many employment contracts have a fixed duration, the employer often reserve the right to change the probation duration and in some jobs, the probation duration is not explicitly written down.

#### 4.1.2 Deferred Compensation

The employment structure in Proposition 2 immediately implies that the compensations of the workers are backloaded. In addition, the expected profit of the employer on the employee decreases over time.

**Corollary 3:** *If the optimal relational contract is implemented by a sequence of stationary contracts once the worker receives permanent employment, then*

(i): *The expected wage of the agent within the employment relationship is nondecreasing over time.*

(ii): *The expected payoff of the principal is nonincreasing over time.*

**Proof.** (i) is immediate. For (ii), the principal's expected payoff is a constant when she's in Phase 2 or Phase 3. When in Phase 1, the principal is always getting  $py - \underline{w}$  per period, which is higher than her normalized payoff. ■

Backloaded compensation has received much attention from economists; see for example Salop and Salop (197X) for a screening argument, Lazear (1981) for an incentive argument, and Harris and Holmstrom (1982) for a learning and insurance argument. The closest to the current paper is Lazear (1981), which also has a model of dynamic moral hazard. The key difference is that in Lazear's model, a) firms can commit to their wage contracts (so there will be mandatory retirement rule) and b) the efforts of the workers are observed perfectly, so there will not be turnovers.

In addition to the backloaded compensation, the model gives more specific prediction on the composition of compensation: the bonus to total compensation ratio increases over employment duration as well. This prediction appears to fit the incentive structure of some occupations. In sales job in particular, it is well-known that the commission rate increases with seniority, see for example Lin (2005) and Coughlan et al (2005). It will be interesting to test this prediction more broadly across occupations using large datasets.

The flip side of the backloaded compensation is that the profitability of the principal over the agent decreases over time. Of course, this prediction can be reversed if we assume that the worker can accumulate firm-specific human capital over time. Nevertheless, it is not impossible that firms may make less money on their more senior employees. For example, Circuit City did a study on the profitability of their salesforce and discovered that their more experienced salespersons actually delivered lower profit.

It should be emphasized that the dynamic incentive studied here implies that "firing workers with the lowest marginal productivity/wage ratio" may not be the optimal firing policy. Firing more senior (and more expensive) workers may be interpreted as a violation of the implicit contract, and this may result in lower efforts from the workers. This appears to be what has happened in Circuit City. Philp Schoonover, who resigned as CEO in Sept 2008, has been criticized for his blunder in firing the more experienced workers:

"Schoonover was slammed for his cost-cutting decision in early 2007 to fire 3,400 of Circuit City's most experienced employees. The company said at the time that they were earning too much money and could be replaced with cheaper workers. But analysts said the move devastated morale and led to a decline in service." Business Week, Sept 22, 2008

### 4.1.3 Turnover Dynamics and Tenure

Due to the uncertainty in output, this model provides rich turnover dynamics. In particular, the turnover rate may not be monotone with respect to the employment duration. In some cases, the optimal relational contract implies a turnover pattern that has features of "tenure": there is a fixed time in which the employee will be terminated if no permanent employment is obtained prior to that.

**Corollary 4:** *When  $\underline{u} + \frac{(1-\delta-\delta q)}{\delta(p-q)}c \geq \underline{w}$ , there exists  $T^*$  such that the turnover rate is 0 for  $t < T^*$  and is again 0 for  $t > T^* + 1$ . Generically, turnover happens only in  $T^*$ .*

**Proof.** When  $\underline{u} + k \geq (\underline{w} - c) + \frac{\delta pk}{(1-\delta)}$ , Corollary 2 gives an explicit formula of  $f$ . There are two cases to consider. In case 1, there's a unique  $u_n$  that maximizes  $f(u)$ . In this case, if any of the output in the first  $n + 2$  periods is positive, the agent receives permanent employment. Otherwise, the agent's continuation payoff moves to  $u_{n+1-t}$  in period  $t$ , and is terminated at time  $t = n + 2$ .

In case 2, there exists  $n$  such that  $f(u)$  is maximized in  $[u_{n-1}, u_n]$ . In this case, if no positive outcome has been generated, the agent's continuation will be in  $[u_{n-t}, u_{n+1-t}]$  in time  $t$ . And the agent will be terminated either in time  $t = n + 1$  or  $t = n + 2$ . ■

In this example, the condition for the parameters is exactly that in Corollary 2. Therefore, the Pareto frontier has countable number of extremal points in this example. The optimal relational contract corresponds to one of the kink ( $u_n$  for some  $n$ ). One success moves the agent to permanent employment. And failure moves the continuation payoff to the kink adjacent to the left until  $(u_0, f(u_0))$ . And termination occurs when failures occurs at  $(u_0, f(u_0))$ .

Note that the condition in the example is more likely to be satisfied when the probability of success is small. This appears to fit the academic labor market fairly well: writing a few home-run papers can bring an assistant professor over the tenure hurdle, and failure to do so within a fixed date leads to termination of the employment contract.

The turnover pattern from the tenure system is an example of (a degenerate) inverse-U shaped turnover patterns with respect to employment duration. The inverse-U shaped turnover pattern appears to be hard to generate theoretically but empirically relevant, see Jovanovic (1979) for a model and Farber (1994) for an empirical investigation. It will be interesting to investigate the turnover patterns for other parameter values of this model. One possibility is to look at the continuous time limit of this discrete time model, and then the turnover pattern may be obtained through simulation or even analytically.

Finally, it should be pointed out that while this model generates rich turnover patterns, there is still a commonality among all possible turnover patterns. The model predicts that, as

time goes to infinity, the turnover rate goes to zero. This is because as time goes to infinity, for a worker remaining in the employment relationship, the probability that he has received permanent employment has approached 1.

#### 4.1.4 Path Dependency

We finish this subsection by noting that there is path dependency in this model. In particular, the agent prefers having a high output followed by a low output than vice versa. Earlier success is more important both in terms of increasing future expected compensation and in terms of reducing the probability of termination. We state the result formally in the next corollary.

**Corollary 5:** *For an agent with expected payoff  $u \in [u_0(\underline{u}), u_e]$ , define  $U_h(u)$  as his continuation payoff following a high output. Also define  $U_{hl}(u)$  and  $U_{lh}(u)$  similarly. If  $U_h(u) \in [u_0(\underline{u}), u_e]$ , then*

$$U_{hl}(u) \geq U_{lh}(u).$$

**Proof.** Direct calculation. ■

The path dependency results from the way the incentive is structured. Since the reward for success in the probation phase in this model is deferred, this implies that the agent will receive a bigger "interest payment" for earlier success. The interest payment takes the form of a higher probability of higher future expected payoffs, and in particular, a lower probability of termination.

This prediction is related with the idea of "fast track" in the internal labor market literature, which states that workers who experience high wage growth in the past are more likely to have high rates of wage growth in the future. More formally, changes of wages are positively serially correlated; see for example Baker, Gibbs, Holmstrom (1994a, b). To explain this phenomenon, the existing literature have relied on heterogeneity in individual abilities; see for example Meyer (1991), Bernhardt (1995), and Gibbons and Waldman (1998).

In this model, while there is no fast track result for wages because there are multiple wage paths consistent with the optimal relational contract once the worker receives permanent employment, Corollary 5 implies a fast track result concerning turnover. In particular, workers who have received good performances are less likely to be terminated. Moreover, the termination is less likely for workers who have good performances earlier. It will be interesting to test this prediction if data on performance evaluations and turnover rates are available. The academic labor market is another example in which this prediction may be tested.

## 4.2 Outside Options, Minimum Wage, and Efficiency

In this subsection, we study how the optimal relational contract is affected by the outside options of the players and by the minimum wage. We show that first, as the agent's outside option improves, the principal's expected payoff decreases, the agent's expected payoff increases, and the overall efficiency increases. Second, as the principal's outside option improves, the principal's expected payoff increases, the agent's expected payoff decreases, and the overall efficiency decreases. Moreover, the aggregate turnover rate of the agent increases. Finally, as the minimum wage increases, the principal's payoff decreases, but the change in agent's payoff is ambiguous.

Our comparative statics results are derived by exploiting the monotonicity of certain functional operators. We think this proof method is of independent interest. Difficulties in obtaining explicit comparative statics results on dynamic incentive problems in discrete time setting is one motivation for studying continuous time models; see for example DeMarzo and Sannikov (2007). Here, comparative statics results can be obtained without knowing the exact value of the optimal PPE payoff. We hope this proof method can be of use for future models of dynamic incentives.

We collect our proofs in the Appendix. We give detailed proofs on the effects of the agent's outside option on the payoffs of the players and the efficiency, and we sketch out the proofs of the effects of the principal's outside option and the minimum wage because the proof method is similar.

We first examine the effect of the agent's outside option. Define  $F(u, \underline{u}, \underline{v}, \underline{w})$  as the Pareto frontier of the PPE payoff when the agent's value is  $u$ , his outside option is  $\underline{u}$ , the principal's outside option is  $\underline{v}$ , and the minimum wage is  $\underline{w}$ . For notational simplicity, we omit  $(\underline{v}, \underline{w})$  and define  $u_0(\underline{u})$  as the unique vector such that  $L(u_0(\underline{u})) = \underline{u}$ . Define  $u^P(\underline{u}, \underline{v}, \underline{w})$  as the principal's maximum payoff is the agent's outside option is  $\underline{u}$ . Define  $u^A(\underline{u}, \underline{v}, \underline{w})$  as the associated payoff of the agent. We have the following results.

**Proposition 3:** *If  $\max_u \{F(u, \underline{u}, \underline{v}, \underline{w})\} > F(u_0(\underline{u}), \underline{u}, \underline{v}, \underline{w})$ , then*

$$\begin{aligned} \frac{\partial u^P(\underline{u}, \underline{v}, \underline{w})}{\partial \underline{u}} &< 0; \\ \frac{\partial u^A(\underline{u}, \underline{v}, \underline{w})}{\partial \underline{u}} &> 0; \\ \frac{\partial (u^P(\underline{u}, \underline{v}, \underline{w}) + u^A(\underline{u}, \underline{v}, \underline{w}))}{\partial \underline{u}} &> 0. \end{aligned}$$

**Proof.** See Appendix. ■

Proposition 3 shows that as the agent's outside option increases, his equilibrium payoff increases, the principal's payoff increases, and the overall efficiency increases. The intuitions are as follows. When the agent's outside option increases, using the threat of terminating the relationship becomes a less effective way for the principal to provide incentive. Therefore, the principal will use more bonus as incentives, and since bonus gives rents to the agent, the principal's payoff decreases. It follows that the optimal relational contract will result in less terminations, and thus the overall efficiency increases. Finally, in this change, the agent's expected payoff increases both because he's less likely to be terminated and because even if he's terminated, his expected payoff is higher.

A similar reasoning explains these comparative statics effects are reversed when the principal's outside option increases. In this case, the principal finds the threat of termination becomes more cost effective. Therefore, the principal will use termination more often, and the efficiency of the relationship decreases even if the payoff of the principal increases. Finally, the agent's payoff decreases because he's more likely to be terminated. Proposition 4 states these results formally.

**Proposition 4:** *If  $\max_u \{F(u, \underline{u}, \underline{v}, \underline{w})\} > F(u_0(\underline{u}), \underline{u}, \underline{v}, \underline{w})$ , then*

$$\begin{aligned} \frac{\partial u^P(\underline{u}, \underline{v}, \underline{w})}{\partial \underline{v}} &> 0; \\ \frac{\partial u^A(\underline{u}, \underline{v}, \underline{w})}{\partial \underline{v}} &< 0; \\ \frac{\partial (u^P(\underline{u}, \underline{v}, \underline{w}) + u^A(\underline{u}, \underline{v}, \underline{w}))}{\partial \underline{v}} &< 0. \end{aligned}$$

**Proof.** See Appendix. ■

A direct consequence of Proposition 4 is that the overall turnover rate increases with the principal's outside option. While the principal's option is treated as an abstract parameter, both local market conditions or the prestige of the employer may be used as proxies. It will be interesting to test this prediction empirically.

**Corollary 6:** *The aggregate turnover probability increases with  $\underline{v}$ .*

**Proof.** This follows from that  $u_0$  does not change with  $\underline{v}$  and the agent's payoff decreases, so a standard coupling argument works. ■

Finally, we examine the effect of the minimum wage. We show that the minimum wage decreases the principal's payoff and the overall efficiency.

**Proposition 5:** *If  $\max_u \{F(u, \underline{u}, \underline{v}, \underline{w})\} > F(u_0(\underline{u}), \underline{u}, \underline{v}, \underline{w})$ , then*

$$\frac{\partial u^P(\underline{u}, \underline{v}, \underline{w})}{\partial \underline{w}} < 0$$

$$\frac{\partial (u^P(\underline{u}, \underline{v}, \underline{w}) + u^A(\underline{u}, \underline{v}, \underline{w}))}{\partial \underline{w}} < 0.$$

**Proof.** See Appendix. ■

The intuition that minimum wage hurts the principal's payoff is straightforward because more rents need to go to the agent's pocket. It follows that termination becomes a relatively more attractive method of inducing incentives because termination is now relatively less costly for the principal. Therefore, the principal will use more of termination as incentive devices, and termination will occur more frequently. This lowers the efficiency of the relationship.

The effect of minimum wage on the agent's expected payoff is ambiguous because there are two forces at work. On the one hand, minimum wage raises the base wage of the agent, and this helps raise his expected payoff. On the other hand, the agent is more likely to be terminated and this damages his welfare. We suspect that if the relationship is less likely to be terminated, then the minimum wage may increase the agent's welfare. On the other hand, if termination is frequent, then minimum wage may actually decrease the agent's payoff.

Finally, we note the following caveat. We derive the welfare property of the minimum wage by holding the principal and the agent's outside options as fixed. In general, the minimum wage may affect the outside options of the players as well. To the extent that the minimum wage may help the agent's outside option, then it may be welfare enhancing. We think that multiple equilibrium is possible and it will be interesting to explore this important issue further.

## 5 Extensions and Discussions

In this section, we conclude the paper by studying properties of the optimal contract when Assumption 1 and 2 fails. We also discuss some possible extensions of the model.

### 5.1 Insufficient Surplus

In this subsection, we study the property of the optimal relational contract when Assumption 1 fails. When

$$py - \frac{[1 - \delta(1 - p)]c}{\delta(p - q)} < \underline{v} + \underline{w},$$

this implies that the efficient outcome (the agent puts in effort in each period) cannot be supported as a PPE. However, the basic structure of the Pareto frontier remains.

In particular, from Lemma 1-Lemma 7, only Lemma 2 fails, in which case we need to determine a new  $u_e$  and the associated  $f(u_e)$ . While there are no explicit formula for  $u_e$ , it can be found as follows.

First, we have the following lemma.

**Lemma 10:** *Suppose  $py - \frac{[1-\delta(1-p)]c}{\delta(p-q)} < \underline{v} + \underline{w}$ , then*

$$f(L(u_e) + k) = \underline{v}.$$

**Proof.** We know that the slope of  $f$  for  $u > u_e$  is negative one. Let  $u_z$  be the point where  $f(u_z) = \underline{v}$ . If  $L(u_e) + k < u_z$ , then we will be able to construct an equilibrium profile that achieves a payoff of  $u_e + \varepsilon$  for the agent and higher than  $f(u_e + \varepsilon)$  for the principal. ■

In other words, if we know  $u_e$ , we know  $f(u_e)$ . Now suppose we let  $u_e = Z$ . This induces an operator via Lemma 6 on the value of  $f(u)$  on  $[u_0, u_e]$ . In equilibrium, we must have the function  $f$  as a fixed point of the operator satisfies that  $f(u_e) = Z$ . Just as the procedure in Lemma 7, there are some monotonicity properties on the operator induced by  $Z$ , and this will help us find  $u_e$ .

Since efficiency cannot be reached in this case, the optimal contract is clearly inefficient. But it may be possible that the optimal contract is constrained optimal, in the sense that  $f(u_e)$  maximizes the PPE payoff of the principal. It will be interesting to explore the necessary and sufficient condition for this.

In terms of employment dynamics, the basic structure here is similar to that in Proposition 2, but we no longer have permanent employment as an absorbing state, so the relationship terminates with probability 1.

**Proposition 6:** *If  $py - \frac{[1-\delta(1-p)]c}{\delta(p-q)} < \underline{v} + \underline{w}$ , then as  $t \rightarrow \infty$ , the relationship dissolves with probability 1.*

**Proof.** It is fairly clear that  $(\underline{u}, \underline{v})$  is the only absorbing state of the relationship. ■

Note that even if there is no permanent employment in this model, it does not mean that the agent never receives bonus payment. In particular if the agent's continuation payoff  $u \geq u_e$ , we can implement the optimal relational contract that the agent will receive a bonus of  $b$  at the beginning of next period if the output is high. The difference is that we must have  $b < k$  because

there isn't sufficient surplus to make the principal incentive compatible to give a bonus of amount  $k$ . But to preserve incentive for the agent to provide effort, we must have the continuation payoff of the agent to drop below  $u$ , whereas in the case sufficient surplus, the incentive can be provided entirely by bonus and no such drop of continuation payoff needs to take place.

## 5.2 Low Minimum Wage

While it appears natural for us to model limited liability by assuming  $\underline{w} \geq \underline{u}$ , we explore how the results changes when  $\underline{w} < \underline{u}$ . We report two results. First, it is clear that for  $\underline{w}$  sufficiently low so that

$$\underline{w} - c + pk \leq \underline{u},$$

limited liability has no bite and we can use stationary contract to provide incentive just as in Levin (2003).

Second, when the minimum wage is not so small so that

$$\frac{\underline{u}}{\delta} \leq u_0 = (1 - \delta)(\underline{w} - c) + \delta(\underline{u} + pk)$$

we can show that  $f$  remains a straight line in  $[\underline{u}, u_0]$ , and we have

$$f(\underline{u}) = \max\{\underline{u}, \frac{\underline{u}f(u_0)}{u_0}\}.$$

In this case, the basic structure of the PPE set remains and the basic structure of the employment dynamics remains.

However, when  $\underline{w}$  is in the intermediate range, we can no longer show that  $f$  is a straight line between  $\underline{u}$  and  $u_0$ , and the analysis becomes considerably more difficult. It will be interesting to explore the property of the optimal relational contract for this range of minimum wage.

## 5.3 Multiple Actions and Outcomes

It maybe possible to extend the model to allow multiple effort level and multiple outcomes. We suspect that multiple outcomes won't change the results that much. This is mainly because in existing models of moral hazard with a risk neutral agent, the optimal contract with limited liability mostly takes a two-step form depending on the likelihood ratio; see for example Levin (2003). But the analysis will be more difficult.

When there are multiple effort levels, I suspect that an additional result we might get is that the workers will be asked to start with low effort level and gradually climb up the effort ladder. If this can be shown, this gives a structure similar to Macleod and Malcomson (1988) even if there is no adverse selection here.

## 5.4 Full Commitment

We study relational contract because explicit long-term contracts appears to be rare in real life. However, it is natural to ask how the employment dynamics in this model changes if we assume that firms can write down explicit contracts and commit to it. It can be shown that, under the two assumptions of the paper (high minimum wage and sufficient surplus), the employment dynamics does not change. When the surplus in the relationship is small, we suspect that full commitment contracts can potentially do better but we suspect that the qualitative patterns of employment dynamics may be similar.

Interestingly, dynamic incentive contracts with limited liability under full commitment have recently received extensive attention from the finance literature; see for example Biais et al (2007), DeMarzo and Fishman (forthcoming), and DeMarzo and Sannikov (2006). In these models, an entrepreneur with limited liability needs to finance a multi-period investment project. Moral hazard arises because the entrepreneur may hide the cash flows from the investors. Using recursive approach, these papers show that the equilibrium Pareto frontier has similar property as that of this paper. In particular, its slope is -1 to the right of a high threshold. And termination can occur to the left of a low threshold. Unlike this paper, the analysis of discrete-time model in that setting appears difficult: it is not known how to solve the Pareto frontier, and no comparative is conducted. Moreover, those papers are interested in how standard securities can be used to implement the optimal contract.

## 5.5 Multiple Projects and Promotion

Another obvious extension of the model is to allow for multiple projects. There are two results readily available for this extension. First, as in the static framework, limited liability may induce the principal to assign "safer" but less efficient projects to the agent.

Second, multiple projects can generate richer employment dynamics depending on the properties of available projects. For example, if there is a safe project which is more efficient than the outside option but less than a risky project, then the punishment to the agent may not be the termination of the relationship, but instead to be assigned to the safe project forever. This helps explain, for example, why the promotion probability of a worker decreases the longer he's at a job, yet he is not necessarily fired or demoted; see for example, Baker, Gibbs, Holmstrom (1994).

## 5.6 Replacement and General Equilibrium

In this model, the parties forever receive their outside options once the relationship is terminated. In reality, the principal may hire another agent after the relationship is terminated, and the agent may find another principal. In a partial equilibrium setting, Fong and Li (2008) examines the

consequence of allowing the principal to replace the current worker. It is shown that the basic structure of the employment dynamics is similar, where the "up-or-out" structure is preserved. On the other hand, the possibility of replacement improves the efficiency of the relationship. In particular, if the principal can replace the current agent without cost, full efficiency may be achieved.

It will also be interesting to allow the impact of replacement on relational contract in a general equilibrium framework. If a safe project is available, we suspect that multiple equilibrium may arise. In one equilibrium, lots of turnover take place and the workers are not assigned to the safe project. In another equilibrium, little turnover take place and workers are assigned the safe project in case of a sequence of bad outputs.

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## 6 Appendix

The appendix has two parts. The first part characterizes the Pareto frontier. The second part collects the proofs for comparative statics.

### 6.1 Characterize the Pareto Frontier

We follow six steps. First, we show that there exists a threshold  $u_e$  such that  $f$  has a slope of  $-1$  to the right of  $u_e$  (Lemma 1). Second, we determine the exact value of  $u_e$  and  $f(u_e)$  (Lemma 2). Third, we show that there exists a threshold  $u_0$  (to be defined below) such that to the left of  $u_0$ ,  $f$  is a straight line between  $(\underline{u}, \underline{v})$  and  $(u_0, f(u_0))$  (Lemma 4). Fourth, we determine the value of  $u_0$  (Lemma 5). Fifth, we show that the value of  $f$  between  $u_0$  and  $u_e$  is the fixed point of a contraction mapping indexed by the value of  $f(u_0)$  (Lemma 6). Finally, we show that  $f(u_0)$  can be found as a fixed point to a monotone contraction mapping.

Denote  $u_e$  as the smallest PPE payoff of the agent that maximizes the sum of the payoff of the principal and agent. First, we show that the Pareto frontier of the PPE past  $u_e$  has a slope of  $-1$ .

**Lemma 1:** For  $u \in [u_e, u_e + f(u_e) - \underline{v}]$ ,

$$f(u) = f(u_e) + u_e - u.$$

**Proof.** Suppose in the PPE that achieves  $(u_e, f(u_e))$ , the principal offers  $w_1$  in period 1. Now consider a different strategy that follows the previous equilibrium except in period 1 the principal offers  $w_1 + u - u_e$ . If the agent rejects the offer in period 1 (or the principal fails to make this offer in period 1), the principal and the agent will believe that and no effort will be put in. It is then easy to check that this strategy profile is a PPE. Moreover, this PPE achieves a payoff of  $(u, f(u_e) + u_e - u)$ . Therefore,  $f(u) \geq f(u_e) + u_e - u$ .

On the other hand, if  $f(u) > f(u_e) + u_e - u$ , this violates the the definition that  $u_e$  maximizes the sum of the principal and the agent's payoff. ■

We next determine the exact value of  $(u_e, f(u_e))$ . In the efficient outcome of the game, effort is put in each period. For the agent to have incentive to put in effort in any period, we must have

$$(1 - \delta)(-c) + \delta((1 - p)u_L + pu_H) \geq \delta((1 - q)u_L + qu_H),$$

where  $u_H$  corresponds to the agent's continuation payoff after a good outcome and  $u_L$  corresponds to the agent's continuation payoff after a bad outcome.

The next lemma shows that the smallest agent's payoff in an efficient equilibrium satisfies

$$u_e = L(u_e).$$

**Lemma 2:**

$$u_e = (\underline{w} - c) + \frac{\delta pk}{(1 - \delta)} = L(u_e).$$

**Proof.** Consider the following strategy: on the equilibrium path: the principal offers the agent  $\underline{w}$  in period 1. The agent accepts and puts in effort. In all future periods, the principal offers the agent  $\underline{w} + k$  if the previous outcome is  $Y = y$  and offers  $\underline{w}$  otherwise. Off the equilibrium path: the agent never puts in effort and the agent never offers a contract to the principal. By Assumption 1, this strategy can be shown to be a PPE, and it achieves a payoff of  $((\underline{w} - c) + \frac{\delta pk}{(1 - \delta)}, py - c - ((\underline{w} - c) + \frac{\delta pk}{(1 - \delta)}))$ . Therefore,  $u_e \leq (\underline{w} - c) + \frac{\delta pk}{(1 - \delta)}$ .

Now if  $u_e < (\underline{w} - c) + \frac{\delta pk}{(1 - \delta)}$ , this implies that  $L(u_e) < u_e$ . Since  $f(u_e) + u_e = py - c$ , this implies that  $L(u_e) + f(L(u_e)) = py - c$  as well. But by the definition of  $u_e$ , we know that  $L(u_e) + f(L(u_e)) < u_e + f(u_e) = py - c$ , so this is a contradiction. ■

Next, we characterize  $f$  to the left of  $u_e$ . The key observation is that the PPE payoff is convex, so  $f$  must be concave. This leads to the next lemma, which shows that in any PPE payoff that obtains the Pareto frontier with  $u \in (\underline{u}, u_e)$ , the first period wage must be the minimum wage.

**Lemma 3:** *For any PPE that reaches  $(u, f(u))$  with  $u \in (\underline{u}, u_e)$ , the first period wage of the agent must be  $w_1 = \underline{w}$ .*

**Proof.** If  $\underline{u} = u_e$ , then there's nothing to prove.

Now let  $\underline{u} < u_e$ . Take  $u \in (\underline{u}, u_e)$ . Suppose in a PPE profile that obtains  $(u, f(u))$ , the first period wage  $w_1 > \underline{w}$ . Now adapt the equilibrium by lowering the first period wage to  $\underline{w}$ . It remains an equilibrium that the agent accepts the period 1 wage and continue the play with the previous equilibrium. (If the agent rejects the period 1 wage, the principal never offers the contract again). This new PPE achieves a payoff of  $(u - (1 - \delta)(w_1 - \underline{w}), f(u) + (1 - \delta)(w_1 - \underline{w}))$ .

Therefore, we have

$$f(u - (1 - \delta)(w_1 - \underline{w})) \geq f(u) + (1 - \delta)(w_1 - \underline{w}).$$

But this says that the slope of  $f$  at  $u$  is smaller than  $-1$ , yet the slope of  $f$  at  $u_e$  is  $-1$ . This violates the concavity of  $f$ . ■

Now define  $u_0$  as the smallest  $u$  in which  $(u, f(u))$  is obtained by requiring the agent to put in effort in period 1. The next lemma shows that  $f(\underline{u}) = \underline{v}$  and  $f$  is a straight line between  $(\underline{u}, \underline{v})$  and  $(u_0, f(u_0))$ .

**Lemma 4:** For  $u \in [\underline{u}, u_0]$ ,

$$f(u) = \underline{v} + \frac{u - \underline{u}}{u_0 - \underline{u}}(f(u_0) - \underline{v}).$$

**Proof.** It is clear that  $(\underline{u}, \underline{v})$  is a PPE payoff, where on the equilibrium path the principal never offers the agent a contract, and off the equilibrium path, the agent never puts in effort. By Assumption 2, we have  $\underline{w} \geq \underline{u}$ , and the assumption that  $qy < \underline{v}$  implies that we must have  $\underline{v} = f(\underline{u})$ .

The convexity of PPE payoff immediately imply that  $f(u) \geq \underline{v} + \frac{u - \underline{u}}{u_0 - \underline{u}}f(u_0)$ . Now suppose the inequality is strict, there are two cases to consider. First, suppose the equilibrium payoff  $(u, f(u))$  is reached by a combination of  $(\underline{u}, \underline{v})$  and  $(u', f(u'))$  for some  $u' \geq u_0$ . In this case, the weak concavity of  $f$  implies that  $(u, f(u))$  cannot lie strictly above the segment formed by  $(\underline{u}, \underline{v})$  and  $(u_0, f(u_0))$ .

In the second case, the equilibrium payoff is reached by a pure play. Now let  $u$  be the largest payoff for the agent such that  $f(u) > \underline{v} + \frac{u - \underline{u}}{u_0 - \underline{u}}f(u_0)$ .<sup>5</sup> From Lemma 3 and the definition of  $u_0$ , we know that the first period play payoff is given by either  $(\underline{u}, \underline{v})$  or  $(\underline{w}, qy)$ . In either case, it is clear that  $(u_0, f(u_0))$  lies strictly below the linear combination of  $(u, f(u))$  and its continuation payoff. This is a contradiction. ■

The next lemma gives the exactly value of  $u_0$ . In particular, we have

$$L(u_0) = \underline{u}.$$

**Lemma 5:**

$$u_0 = (1 - \delta)(\underline{w} - c) + \delta(\underline{u} + pk)$$

**Proof.** It is clear  $L(u_0) \geq \underline{u}$ , which is the agent's maxmin payoff. Now if  $L(u_0) > \underline{u}$ , we argue that there exists a PPE payoff that gives the agent the payoff of  $u_0 - \varepsilon$  and the principal a payoff that lies on the line segment between  $(\underline{u}, \underline{v})$  and  $(u_0, f(u_0))$ , and this violates the definition of  $u_0$ . In particular, let  $s$  be the slope between  $(\underline{u}, \underline{v})$  and  $(u_0, f(u_0))$ . Then by the weak concavity of  $f$ , we know that both  $(L(u_0) - \varepsilon, f(L(u_0)) - s\varepsilon)$  and  $(L(u_0) + k - \varepsilon, f(L(u_0) + k) - s\varepsilon)$  are PPE payoffs. And strategy profile that pays the agent  $w_1 = \underline{w}$ , requires the agent to put in effort in period 1, and promise the agent with the above two continuation payoffs (given the

<sup>5</sup>If no such points exist, take one close enough to the limsup.

output as  $y$  or 0), will be an equilibrium that gives payoff of  $(u_0 - \varepsilon, f(u_0) - s\varepsilon)$ . This proves that  $L(u_0) = \underline{u}$ . ■

Note that we have determined the shape of  $f$  to the left of  $u_0$  and to the right of  $u_0$ , we are ready to determine the value of  $f$  between these two points. The next lemma gives such a functional equation.

**Lemma 6:** For  $u \in [u_0, u_e]$ ,

$$f(u) = (1 - \delta)(py - \underline{w}) + \delta[pf(L(u) + k) + (1 - p)f(L(u))]. \quad (8)$$

**Proof.** This follows in two steps. The first step shows that for  $u \in [u_0, u_e]$ ,  $(u, f(u))$  can be obtained by an equilibrium profile in which the first period play requires effort. Now suppose the contrary. Let  $u_l$  be the largest point such that  $(u_l, f(u_l))$  lies on the line given by  $(\underline{u}, \underline{v})$  and  $(u_0, f(u_0))$ . Suppose  $u \in [u_l, u_e]$ . There are two cases to consider.

In the first case,  $(u, f(u))$  can be reached by a pure play in period 1. In this case, by Lemma 3, the first period play payoff is given by either  $(\underline{u}, \underline{v})$  or  $(\underline{w}, qy)$ . In either case, the slope between  $(u, f(u))$  and its continuation payoff will exceed the slope between  $(\underline{u}, \underline{v})$  and  $(u_0, f(u_0))$ . This violates the concavity of  $f$ .

In the second case, the  $(u, f(u))$  is reached through a mix. Since  $(u, f(u))$  lives in a two dimension space and  $f$  is concave, we may assume

$$(u, f(u)) = p_1((u_1, f(u_1))) + (1 - p_1)((u_2, f(u_2)))$$

for some  $p_1, u_1,$  and  $u_2$ , where  $(u_1, f(u_1))$  and  $(u_2, f(u_2))$  are reached through pure play in period 1. If  $(u_i, f(u_i)), i = 1, 2$  has first period play that doesn't require effort, the previous paragraph implies that  $u_i \notin [u_l, u_e]$ . But then the concavity of  $f$  implies that  $(u, f(u))$  can be obtained by linear combination of points with the agent's payoff belonging to  $[u_l, u_e]$ .

To finish the first step, if  $u \in (u_0, u_l)$ , then  $(u, f(u))$  can be achieved by mixing  $(u_0, f(u_0))$  and  $(u_l, f(u_l))$ . And since both  $(u_0, f(u_0))$  and  $(u_l, f(u_l))$  can be achieved by requiring effort in period 1, so can  $(u, f(u))$ .

In the second step, we note that the equation follows because to achieve the maximum payoff for the principal, a) the continuation payoff must lie on the Pareto frontier, and b) the distance of payoff between the good and bad outcomes for the agent needs not to exceed  $k$  by the concavity of  $f$ . ■

Lemma 6 motivates us to define the following operator. Let  $g$  be a bounded function on  $[u_0, u_e]$ . We define  $T_Z$  as follows:

$$\begin{aligned}
T_Z g(u) &= (1 - \delta)(py - \underline{w}) \\
&\quad + \delta p(1_{\{L(u)+k < u_e\}}g(L(u) + k) + 1_{\{L(u)+k \geq u_e\}}(py - c - (L(u) + k))) \\
&\quad + \delta(1 - p)(1_{\{L(u) < u_0\}}(\underline{v} + \frac{L(u) - \underline{u}}{u_0 - \underline{u}}(Z - \underline{v}) + 1_{\{L(u) \geq u_0\}}g(L(u))).
\end{aligned}$$

**Lemma 7:**

$$0 \leq \frac{dg_Z(u_0)}{dZ} < 1.$$

**Proof.** There are two cases to consider. First, suppose  $L(u_0) + k \geq u_e$ . In this case, the result is immediate because

$$g_Z(u_0) = (1 - \delta)(py - \underline{w}) + \delta((1 - p)\underline{v} + pf(L(u_0) + k))$$

And since when  $L(u_0) + k \geq u_e$ ,  $f(L(u_0) + k)$  is independent of  $Z$ , we have

$$\frac{dg_Z(u_0)}{dZ} = 0.$$

Now suppose  $L(u_0) + k < u_e$ . Take the unique fixed point of  $Tz$ ,  $g_Z$ , and note that

$$T_{Z+\varepsilon}(g_Z + \varepsilon) \leq T_Z g_Z + \delta\varepsilon \leq T_Z g_Z + \varepsilon = g_Z + \varepsilon.$$

Since  $T$  is monotone (i.e.  $Tg_1 \leq Tg_2$  if  $g_1 \leq g_2$  where  $g_1 \leq g_2$  iff  $g_1(u) \leq g_2(u)$  for all  $u$ ), we have

$$T_{Z+\varepsilon}^2(g_Z + \varepsilon) \leq T_{Z+\varepsilon}^1(g_Z + \varepsilon),$$

and more generally

$$T_{Z+\varepsilon}^n(g_Z + \varepsilon) \leq T_{Z+\varepsilon}^{n-1}(g_Z + \varepsilon) \leq \dots \leq g_Z + \varepsilon.$$

Since  $g_{Z+\varepsilon} = \lim_{n \rightarrow \infty} T_{Z+\varepsilon}^n(g_Z + \varepsilon)$ , we have

$$g_{Z+\varepsilon} \leq g_Z + \varepsilon.$$

In particular, this implies that

$$g_{Z+\varepsilon}(L(u_0) + k) \leq g_Z(L(u_0) + k) + \varepsilon.$$

Finally, we have

$$g_Z(u_0) = (1 - \delta)py + \delta((1 - p)x + pg_Z(L(u_0) + k)).$$

$$g_{Z+\varepsilon}(u_0) = (1 - \delta)py + \delta((1 - p)x + pg_{Z+\varepsilon}(L(u_0) + k)).$$

Therefore,

$$g_{Z+\varepsilon}(u_0) - g_Z(u_0) = \delta p [g_{Z+\varepsilon}(L(u_0) + k) - g_Z(L(u_0) + k)] \leq \delta p \varepsilon$$

and thus

$$\frac{dg_Z(u_0)}{dZ} = \lim_{\varepsilon \rightarrow \infty} \frac{g_{Z+\varepsilon}(u_0) - g_Z(u_0)}{\varepsilon} \leq \delta p < 1.$$

On the other hand, it can be checked that

$$T_{Z+\varepsilon}(g_Z) \geq g_Z,$$

so a similar reasoning as above gives that

$$\frac{dg_Z(u_0)}{dZ} \geq 0.$$

■

Lemma 7 immediately implies that there will be at most one value  $f(u_0)$  that satisfies (6). The existence of such value is guaranteed by the fact that  $f$  is properly defined and it must satisfy (6).

An more direct approach to prove the existence is as follows. In the proof of Lemma 7, we know that  $g_Z$  is weakly increasing in  $Z$ . In addition, it can be shown that the  $g_Z$  induced by  $T_Z$  is nonexpansive in the sense that

$$\|g_{Z+\varepsilon} - g_Z\| \leq \varepsilon.$$

It can also be shown that a weakly increasing non-expansive map has a unique fixed point (see the proof of Lemma 9 for a formal proof), so there is a unique point  $Z^*$  such that

$$Z^* = g_{Z^*}(u_0).$$

And we have

$$f(u_0) = Z^*.$$

## 6.2 Comparative Statics

In this subsection, we derive comparative statics by exploiting properties of functional operators. The first lemma shows that, in the range of agent's payoffs where efforts are called for, the PPE frontier shrinks continuously as the agent's outside option improves.

**Lemma 8:** If  $\max_u \{F(u, \underline{u}, \underline{v}, \underline{w})\} > F(u_0(\underline{u}), \underline{u}, \underline{v}, \underline{w})$ , then all  $u \in [u_0(\underline{u}), u_e]$

$$-s < \frac{\partial F(u, \underline{u}, \underline{v}, \underline{w})}{\partial \underline{u}} < 0,$$

where  $s$  is the left derivative (with respect to  $u$ ) of  $F(u_0(\underline{u}), \underline{u}, \underline{v}, \underline{w})$ .

**Proof.** Suppose we increase the agent's outside option from  $\underline{u}$  to  $\underline{u} + \varepsilon$ . To simplify notation, suppose  $f_1$  is the PPE Pareto frontier with respect to  $\underline{u}$  defined on  $[u_0(\underline{u} + \varepsilon), u_e]$ . And let  $f_2$  be the PPE Pareto frontier with respect to  $\underline{u} + \varepsilon$  defined on  $[u_0(\underline{u} + \varepsilon), u_e]$ .

We know that  $u_0(\underline{u} + \varepsilon) = u_0(\underline{u}) + \delta\varepsilon$ . We also know that  $u^A(\underline{u} + \varepsilon) \in (u_0(\underline{u} + \varepsilon), u_e)$ . Now take a line segment between  $(\underline{u} + \varepsilon, \underline{v})$  and  $(u_0(\underline{u} + \varepsilon), f_1(u_0(\underline{u} + \varepsilon)))$ . This line segment lies strictly below  $f_1$  (except at the right end point where the two are equal.).

Now define an operator  $T_1$  on functions on  $[u_0(\underline{u} + \varepsilon), u_e]$  as follows.

$$\begin{aligned} T_1 g(u) &= (1 - \delta)(py - \underline{w}) \\ &+ \delta p(1_{\{L(u)+k < u_e\}} g(L(u) + k) + 1_{\{L(u)+k \geq u_e\}} (py - c - (L(u) + k))) \\ &+ \delta(1 - p)(1_{\{L(u) \geq u_0(\underline{u} + \varepsilon)\}} g(L(u) + \\ &1_{\{L(u) < u_0(\underline{u} + \varepsilon)\}} (\underline{v} + \frac{L(u) - (\underline{u} + \varepsilon)}{u_0(\underline{u} + \varepsilon) - (\underline{u} + \varepsilon)} (f_1(u_0(\underline{u} + \varepsilon) - \underline{v}))) \end{aligned}$$

Note that this operator is very similar to the operator  $T$  in Lemma 7, except the "straight line on the left" is smaller than that in  $T$ . It follows that

$$T_1 g \leq Tg \text{ for all } g.$$

In addition,  $T_1$  can be checked to be monotone.

Define  $g_1$  on  $[u_0(\underline{u} + \varepsilon), u_e]$  by  $g_1(u) = f_1(u)$ . Then it is clear that

$$T_1 g_1(u) \leq Tg_1(u) \leq g_1(u).$$

Since  $T$  is monotone, it follows that

$$g_1(u) \geq g_{T_1}(u),$$

where  $g_{T_1}$  is the fixed point of  $T_1$ . From Lemma 7, this implies that  $f_2(u_0(\underline{u} + \varepsilon)) = g_{T_1}(u_0(\underline{u} + \varepsilon)) < g_1(u_0(\underline{u} + \varepsilon)) = f_1(u_0(\underline{u} + \varepsilon))$ .

Now define the operator  $T_2$  on functions on  $[u_0(\underline{u} + \varepsilon), u_e]$  as follows.

$$\begin{aligned}
T_2g(u) &= (1 - \delta)(py - \underline{w}) \\
&+ \delta p(1_{\{L(u)+k < u_e\}}g(L(u) + k) + 1_{\{L(u)+k \geq u_e\}}(py - c - (L(u) + k))) \\
&+ \delta(1 - p)(1_{\{L(u) \geq u_0(\underline{u} + \varepsilon)\}}g(L(u) \\
&+ 1_{\{L(u) < u_0(\underline{u} + \varepsilon)\}}(\underline{v} + \frac{L(u) - (\underline{u} + \varepsilon)}{u_0(\underline{u} + \varepsilon) - (\underline{u} + \varepsilon)}(f_2(u_0(\underline{u} + \varepsilon) - \underline{v})))
\end{aligned}$$

Define  $g_2$  on  $[u_0(\underline{u} + \varepsilon), u_e]$  by  $g_2(u) = f_2(u)$ . We know that  $g_2$  is a fixed point of  $T_2$ . On the other hand, we can check that  $T_2g_1(u) \geq g_1(u)$ . It immediately follows (as in Lemma 7) that  $g_1(u) \geq g_2(u)$  for all  $u$ . Moreover, we note that  $g_1(u) > g_2(u)$  for some neighborhood of  $u_0(\underline{u} + \varepsilon)$ . And since all of  $u < u_e$  reaches the neighborhood of with positive probability, so we have

$$f_2(u) = g_2(u) < g_1(u) = f_1(u) \text{ for all } u \in [u_0(\underline{u} + \varepsilon), u_e].$$

And  $f_2(u_e) = f_1(u_e)$ . This proves the first part of the lemma.

For the second part, suppose the slope between  $(\underline{u}, \underline{v})$  and  $(u_0(\underline{u}), f_1(u_0(\underline{u})))$  is  $s$ , where  $f_1(u_0(\underline{u}))$  is the maximal PPE payoff of the principal at  $u_0(\underline{u})$  assuming that the outside option is  $\underline{u}$ . Take a small  $\varepsilon$ , construct a line through  $(\underline{u} + \varepsilon, \underline{v})$  with slope  $s$ . For small enough  $\varepsilon$ , we can show that this line lies strictly below  $f_1$  from  $\underline{u}$  till  $u_0(\underline{u} + \varepsilon)$ . This is because  $\max_u \{F(u, \underline{u}, \underline{v}, \underline{w})\} > F(u_0(\underline{u}), \underline{u}, \underline{v}, \underline{w})$ , the right derivative of  $f_1$  at  $u_0(\underline{u})$  is strictly positive, so there exists a small  $\varepsilon_1$  such that  $f_1(u_0(\underline{u}) + \varepsilon_1) \geq f_1(u_0(\underline{u}))$ . Now take  $\varepsilon = \varepsilon_1/\delta$  will work.)

Let  $d = s\varepsilon$ . Define an operator  $T_3$  on functions defined on  $[u_0(\underline{u} + \varepsilon), u_e]$ , such that the "straight line on the left" is given by the line through  $(\underline{u} + \varepsilon, \underline{v})$  with slope  $s$ . Define  $g_1$  on  $[u_0(\underline{u} + \varepsilon), u_e]$  by  $g_1(u) = f_1(u)$ .

$$\begin{aligned}
&T_3(g_1(u) - d) \\
&= (1 - \delta)py \\
&+ \delta p(1_{\{L(u)+k < u_e\}}(g_1(L(u) + k) - d) + 1_{\{L(u)+k \geq u_e\}}(py - c - (L(u) + k) - d)) \\
&+ \delta(1 - p)(1_{\{L(u) \geq u_0(\underline{u} + \varepsilon)\}}(g_1(L(u) - d) + 1_{\{L(u) < u_0(\underline{u} + \varepsilon)\}}(\underline{v} + s(L(u) - \underline{u} - \varepsilon))).
\end{aligned}$$

For small enough  $\varepsilon$ , we see that

$$T_3(g_1(u) - d) \simeq g_1(u) - \delta d \geq g_1(u) - d.$$

Now by the uniqueness proof (and that the slope of  $f_1$  is smaller than  $s$  for  $u > u_0(\underline{u} + \varepsilon)$ ), we see that

$$f_2(u_0(\underline{u} + \varepsilon)) > f_1(u_0(\underline{u} + \varepsilon)) - d.$$

Finally, follow the proof procedure in the first part of the theorem, we see immediately that

$$f_2(u) > f_1(u) - d \text{ for all } u \in [u_0(\underline{u} + \varepsilon), u_e].$$

■

The next lemma shows that, while the value of the PPE set shrinks for  $u \in [u_0(\underline{u}), u_e]$ , its slope increases.

**Lemma 9:** *If  $\max_u \{F(u, \underline{u}, \underline{v}, \underline{w})\} > F(u_0(\underline{u}), \underline{u}, \underline{v}, \underline{w})$ , then for almost all  $u \in [u_0(\underline{u}), u_e]$*

$$\frac{\partial^2 F(u, \underline{u}, \underline{v}, \underline{w})}{\partial \underline{u} \partial u} > 0.$$

**Proof.** Again to simplify notation, suppose  $f_1$  is the PPE Pareto frontier with respect to  $\underline{u}$  defined on  $[u_0(\underline{u} + \varepsilon), u_e]$ . And let  $f_2$  be the PPE Pareto frontier with respect to  $\underline{u} + \varepsilon$  defined on  $[u_0(\underline{u} + \varepsilon), u_e]$ . Now define  $h_1 = f'_1$ ; and  $h_2 = f'_2$ . When  $f'_1$  and  $f'_2$  are not properly defined, we use the right limit.

Lemma 8 implies that

$$h_2 > h_1 \quad \text{for } u < u_0(\underline{u} + \varepsilon).$$

Now define

$$\begin{aligned} \tilde{T}(h) &= p(1_{\{L(u)+k < u_e\}}(h) - 1_{\{L(u)+k \geq u_e\}}) \\ &\quad + (1-p)(1_{\{L(u) \geq u_0(\underline{u}+\varepsilon)\}}h(L(u)) + s_1 1_{\{L(u) < u_0(\underline{u}+\varepsilon)\}}), \end{aligned}$$

where  $s_1$  is the slope between  $(\underline{u} + \varepsilon, \max\{x - (\underline{u} + \varepsilon), 0\})$  and  $(u_0(\underline{u} + \varepsilon), f_2(u_0(\underline{u} + \varepsilon)))$ . It is easy to see that

$$\tilde{T}(h_1) \geq h_1.$$

Let us define that

$$h^* = \lim_n \tilde{T}^n(h_1).$$

Since the operator is monotone,

$$\tilde{T}(h^*) = \tilde{T}(\lim_n \tilde{T}^n(h_1)) \geq \tilde{T}(\tilde{T}^n(h_1)) \quad \text{for all } n,$$

so we have  $\tilde{T}(h^*) \geq h^*$ .

On the other hand, take any  $u$ , we know that

$$\begin{aligned}
\tilde{T}(h^*(u)) &= p(1_{\{L(u)+k < u_e\}}(h^*(u)) - 1_{\{L(u)+k \geq u_e\}}) \\
&\quad + (1-p)(1_{\{L(u) \geq u_0(\underline{u}+\varepsilon)\}}h^*(L(u)) + s_1 1_{\{L(u) < u_0(\underline{u}+\varepsilon)\}}) \\
&\leq p(1_{\{L(u)+k < u_e\}}(\tilde{T}^n(h_1(u)) + \varepsilon) - 1_{\{L(u)+k \geq u_e\}}) \\
&\quad + (1-p)(1_{\{L(u) \geq u_0(\underline{u}+\varepsilon)\}}\tilde{T}^n(h_1 L(u)) + \varepsilon) + s_1 1_{\{L(u) < u_0(\underline{u}+\varepsilon)\}} \\
&\leq \tilde{T}(h^*(u)) + \varepsilon.
\end{aligned}$$

And therefore,

$$\tilde{T}h^* = h^*.$$

Moreover, while  $\tilde{T}$  isn't a contraction mapping, it is nevertheless non-expansive, in the sense that

$$\|\tilde{T}h_1 - \tilde{T}h_2\| \leq \|h_1 - h_2\|$$

and it can be checked that in this case, it has a unique fixed point.

To see this, suppose  $h_a$  and  $h_b$  are two fixed points of  $\tilde{T}$ . And let  $M = \|h_a - h_b\|$ . Take the smallest  $u^*$  such that  $\|h_a(u^*) - h_b(u^*)\| = M^6$ . Then we see that  $\|\tilde{T}h_a(u^*) - \tilde{T}h_b(u^*)\| < M$  by the definition of  $\tilde{T}$  unless  $M = 0$ .

Now, this implies that

$$h^*(u) = h_2(u) = f'_2(u).$$

And therefore, we have

$$f'_1(u) \leq f'_2(u)$$

for all  $u \in [u_0(\underline{u} + \varepsilon), u_e]$ . ■

With these two lemmas, we are ready to study the effect of the agent's outside option on the optimal relational contract.

Define  $u^P(\underline{u}, \underline{v}, \underline{w})$  as the principal's maximum payoff is the agent's outside option is  $\underline{u}$ . Define  $u^A(\underline{u}, \underline{v}, \underline{w})$  as the associated payoff of the agent. We have the following results.

---

<sup>6</sup>Note that such  $V^*$  may not exist both because of the achievability of the sup norm and because of the liminf of such  $V^*$ . In this case, we can take the appropriate approximation.

**Proposition 3:** *If  $\max_u \{F(u, \underline{u}, \underline{v}, \underline{w})\} > F(u_0(\underline{u}), \underline{u}, \underline{v}, \underline{w})$ , then*

$$\begin{aligned}\frac{\partial u^P(\underline{u}, \underline{v}, \underline{w})}{\partial \underline{u}} &< 0; \\ \frac{\partial u^A(\underline{u}, \underline{v}, \underline{w})}{\partial \underline{u}} &> 0; \\ \frac{\partial(u^P(\underline{u}, \underline{v}, \underline{w}) + u^A(\underline{u}, \underline{v}, \underline{w}))}{\partial \underline{u}} &> 0.\end{aligned}$$

**Proof.** The first inequality follows directly from Lemma 8.

Lemma 9 implies that  $\frac{\partial u^A(\underline{u}, \underline{v}, \underline{w})}{\partial \underline{u}} \geq 0$ . (It does not imply a strict inequality directly because  $f$  may have kinks at  $u^A$ .)

Moreover, Berger's continuity theorem implies that  $u^A(\underline{u}, \underline{v}, \underline{w})$  is continuous with respect to  $\underline{u}$ . Again to simplify notation, suppose  $f_1$  is the PPE Pareto frontier with respect to  $\underline{u}$  defined on  $[u_0(\underline{u}) + \varepsilon, u_e]$ . And let  $f_2$  be the PPE Pareto frontier with respect to  $\underline{u} + \varepsilon$  defined on  $[u_0(\underline{u}) + \varepsilon, u_e]$ . It then suffices to show that, for small enough  $\varepsilon$ ,  $f_2(u^A(\underline{u}) + \varepsilon) \geq f_1(u^A(\underline{u})) - \varepsilon$ .

To this end, define

$$g_1(u) = f_1(u - \varepsilon) - \varepsilon, \quad \text{for } u \in [u_0(\underline{u}) + \varepsilon, u_e].$$

Now link the line segment between  $(\underline{u} + \varepsilon/\delta, \underline{v})$  and  $(u_0(\underline{u}) + \varepsilon, f_1(u_0(\underline{u})))$ , and define an operator  $T$  on bounded functions between  $[u_0(\underline{u}) + \varepsilon, u_e]$  as

$$\begin{aligned}T(g(u)) &= (1 - \delta)(py - \underline{w}) \\ &+ \delta p(1_{\{L(u)+k < u_e\}}(g(L(u) + k)) + 1_{\{L(u)+k \geq u_e\}}(py - c - (L(u) + k))) \\ &+ \delta(1 - p)(1_{\{L(u) \geq u_0(\underline{u}) + \varepsilon\}}g(L(u)) + \\ &1_{\{L(u) < u_0(\underline{u}) + \varepsilon\}}(\underline{v} + \frac{L(u) - (\underline{u} - \varepsilon/\delta)}{u_0(\underline{u}) + \varepsilon - (\underline{u} - \varepsilon/\delta)}(f_1(u_0(\underline{u})) - \underline{v})),\end{aligned}$$

With this operator, we can show that

$$T(f_1(u)) > f_1(u), \quad \text{for } u \in [u_0(\underline{u}) + \varepsilon, u_e].$$

Now notice that, for  $u \in [u_0(\underline{u}) + \varepsilon, u_e)$ ,

$$\begin{aligned}
T(g_1(u)) &= (1 - \delta)(py - \underline{w}) \\
&\quad + \delta p(1_{\{L(u)+k < u_e\}}(g_1(L(u) + k)) + 1_{\{L(u)+k \geq u_e\}}(py - c - (L(u) + k))) \\
&\quad + \delta(1 - p)(1_{\{L(u) \geq u_0(\underline{u}) + \varepsilon\}}g_1(L(u))) \\
&\quad + 1_{\{L(u) < u_0(\underline{u}) + \varepsilon\}}(\underline{v} + \frac{L(u) - (\underline{u} - \varepsilon/\delta)}{u_0(\underline{u}) + \varepsilon - (\underline{u} - \varepsilon/\delta)}(f_1(u_0(\underline{u})) - \underline{v})) \\
&= (1 - \delta)(py - \underline{w}) \\
&\quad + \delta p(1_{\{L(u)+k < u_e\}}(f_1(L(u) - \delta\varepsilon) + k) - \varepsilon) + 1_{\{L(u)+k \geq u_e\}}(py - c - (L(u) + k)) \\
&\quad + \delta(1 - p)(1_{\{L(u) \geq u_0(\underline{u}) + \varepsilon\}}f_1(L(u) - \delta\varepsilon) - \varepsilon) \\
&\quad + 1_{\{L(u) < u_0(\underline{u}) + \varepsilon\}}(\underline{v} + \frac{L(u) - (\underline{u} - \varepsilon/\delta)}{u_0(\underline{u}) + \varepsilon - (\underline{u} - \varepsilon/\delta)}(f_1(u_0(\underline{u})) - \underline{v})) \\
&\geq T(f_1(u - \delta\varepsilon)) - \delta\varepsilon \\
&> f_1(u - \delta\varepsilon) - \delta\varepsilon \\
&\approx f_1(u) - (f_1'(u) + 1)\delta\varepsilon \\
&= f_1(u) - (f_1'(u) + 1)\varepsilon + (f_1'(u) + 1)(1 - \delta)\varepsilon \\
&\approx f_1(u - \varepsilon) - \varepsilon + (f_1'(u) + 1)(1 - \delta)\varepsilon \\
&= g_1(u) + (f_1'(u) + 1)(1 - \delta)\varepsilon.
\end{aligned}$$

The last inequality follows from the fact that  $f_1'(u) > -1$  for  $u \in [u_0(\underline{u}) + \varepsilon, u_e)$ , which is a consequence of the concavity of  $f$  and the definition of  $u_e$ .

By a similar proof method as in Lemma 8, this implies that for  $u \in [u_0(\underline{u}) + \varepsilon, u_e)$ ,

$$\begin{aligned}
f_2(u) &> g_1(u) + (f_1'(u) + 1)(1 - \delta)\varepsilon \\
&= f_1(u - \varepsilon) - \varepsilon + (f_1'(u) + 1)(1 - \delta)\varepsilon
\end{aligned}$$

This implies that the surplus is strictly increasing.

Finally, from this result and that  $\frac{\partial u^P(\underline{u}, \underline{v}, \underline{w})}{\partial \underline{u}} < 0$ , we have that  $\frac{\partial u^A(\underline{u}, \underline{v}, \underline{w})}{\partial \underline{u}} > 0$ . ■

Similar method as above can establish the following result.

**Proposition 4:** *If  $\max_u \{F(u, \underline{u}, \underline{v}, \underline{w})\} > F(u_0(\underline{u}), \underline{u}, \underline{v}, \underline{w})$ , then*

$$\begin{aligned}
\frac{\partial u^P(\underline{u}, \underline{v}, \underline{w})}{\partial \underline{v}} &> 0; \\
\frac{\partial u^A(\underline{u}, \underline{v}, \underline{w})}{\partial \underline{v}} &< 0; \\
\frac{\partial(u^P(\underline{u}, \underline{v}, \underline{w}) + u^A(\underline{u}, \underline{v}, \underline{w}))}{\partial \underline{v}} &< 0.
\end{aligned}$$

**Proof.** The proof parallels the arguments in Proposition 3, and it is easier. so we omit it here. Neither  $u_0$  and  $u_e$  changes with  $\underline{v}$ , but the value of  $f$  to the left of  $u_0$  increases because of the increase in  $\underline{v}$ . It follows in an argument like Lemma 8 that the value of  $f$  in  $[u_0, u_e]$  must increase so that the principal's payoff must increase. Moreover, an argument similar to Lemma 9 shows that the derivative of  $f$  in  $(u_0, u_e)$  must decrease, so the expected payoff of the agent decreases. Finally, following Proposition, we can define a function  $g(u) = f(u + \varepsilon) + \varepsilon$ , and a similar argument can show that the overall efficiency decreases. ■

Finally, we examine the effect of the minimum wage. We show that the minimum wage decreases the principal's payoff and the overall efficiency.

**Proposition 5:** *If  $\max_u \{F(u, \underline{u}, \underline{v}, \underline{w})\} > F(u_0(\underline{u}), \underline{u}, \underline{v}, \underline{w})$ , then*

$$\begin{aligned} \frac{\partial u^P(\underline{u}, \underline{v}, \underline{w})}{\partial \underline{w}} &< 0 \\ \frac{\partial(u^P(\underline{u}, \underline{v}, \underline{w}) + u^A(\underline{u}, \underline{v}, \underline{w}))}{\partial \underline{w}} &< 0. \end{aligned}$$

**Proof.** Suppose the minimum wage increases from  $\underline{w}$  to  $\underline{w} + \varepsilon$ . In this case, both  $u_0$  and  $u_e$  increases by  $\frac{1-\delta}{\delta}\varepsilon$ . Then the first inequality can be established by using a method similar to Lemma 8 and noting that  $f' \geq -1$ . The second inequality follows Proposition 3 closely. Again to simplify notation, suppose  $f_1$  is the PPE Pareto frontier with respect to  $\underline{w}$  defined on  $[u_0(\underline{u}) + \frac{1-\delta}{\delta}\varepsilon, u_e + \frac{1-\delta}{\delta}\varepsilon]$ . And let  $f_2$  be the PPE Pareto frontier with respect to  $\underline{w} + \varepsilon$  defined on  $[u_0(\underline{u}) + \frac{1-\delta}{\delta}\varepsilon, u_e + \frac{1-\delta}{\delta}\varepsilon]$ . It then suffices to show that, for small enough  $\varepsilon$ ,  $f_2(u^A(\underline{u}) + \varepsilon) \geq f_1(u^A(\underline{u})) - \varepsilon$ . (If  $u^A$  decreases, we immediately have the decrease in efficiency because both the principal and the agent's payoff decrease.)

To this end, define

$$g_1(u) = f_1(u - \varepsilon) - \varepsilon, \quad \text{for } u \in [u_0(\underline{u}) + \frac{1-\delta}{\delta}\varepsilon, u_e + \frac{1-\delta}{\delta}\varepsilon].$$

Now link the line segment between  $(\underline{u}, \underline{v})$  and  $(u^A(\underline{u}) + \frac{1-\delta}{\delta}\varepsilon, f_1(u_0(\underline{u}) + \frac{1-\delta}{\delta}\varepsilon))$ , and define an

operator  $T$  on bounded functions between  $[u_0(\underline{u}) + \frac{1-\delta}{\delta}\varepsilon, u_e + \frac{1-\delta}{\delta}\varepsilon]$  as

$$\begin{aligned}
T(g(u)) &= (1 - \delta)(py - \underline{w} - \varepsilon) \\
&+ \delta p(1_{\{L(u)+k-\frac{1-\delta}{\delta}\varepsilon < u_e\}}(g(L(u) + k - \frac{1-\delta}{\delta}\varepsilon) + \\
&1_{\{L(u)+k-\frac{1-\delta}{\delta}\varepsilon \geq u_e\}}(py - c - (L(u) + k - \frac{1-\delta}{\delta}\varepsilon))) \\
&+ \delta(1 - p)(1_{\{L(u)-\frac{1-\delta}{\delta}\varepsilon \geq u_0(\underline{u})+\frac{1-\delta}{\delta}\varepsilon\}}g(L(u) - \frac{1-\delta}{\delta}\varepsilon) + \\
&1_{\{L(u)-\frac{1-\delta}{\delta}\varepsilon \geq u_0(\underline{u})+\frac{1-\delta}{\delta}\varepsilon\}}(\underline{v} + \frac{L(u) - \frac{1-\delta}{\delta}\varepsilon - \underline{u}}{u_0(\underline{u}) + \frac{1-\delta}{\delta}\varepsilon - \underline{u}}(f_1(u_0(\underline{u}) + \frac{1-\delta}{\delta}\varepsilon) - \underline{v})),
\end{aligned}$$

With this operator, we can show that

$$T(f_1(u)) > f_1(u), \text{ for } u \in [u_0(\underline{u}) + \frac{1-\delta}{\delta}\varepsilon, u_e + \frac{1-\delta}{\delta}\varepsilon].$$

Finally, as in Proposition 3, we have the following:

$$\begin{aligned}
T(g_1(u)) &\leq T(f_1(u - \delta\varepsilon)) - \delta\varepsilon - (1 - \delta)\varepsilon \\
&\quad - (1 - \delta)\varepsilon(pf_1'(u - \delta\varepsilon + k) + (1 - p)f_1'(u - \delta\varepsilon)) \\
&< f_1(u - \delta\varepsilon) - \delta\varepsilon - (1 - \delta)\varepsilon - (1 - \delta)\varepsilon f_1'(u) \\
&\approx f_1(u) - (f_1'(u) + 1)\delta\varepsilon - (1 - \delta)\varepsilon - (1 - \delta)\varepsilon f_1'(u) \\
&= g_1(u).
\end{aligned}$$

■