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A Theory of Player Turnover in Repeated Games

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Abstract

We propose a theory of player turnover in long-term relationships according to which replacement of players suspected of deviation with new players mitigates inefficiency arising from imperfect monitoring even if doing so requires compensating the suspected deviators adequately for them to leave voluntarily. Our theory encompasses turnover allowing and disallowing payment to the departing player, voluntary and involuntary turnover. Through a general model and three applications we establish the following findings. First, player turnover creates Pareto improvement even if the new player is an identical replica of the existing player, so improvement in performance following player change may not be attributable to difference in player abilities. Second, bargaining power of the incumbent player limits the Pareto improvement of and causes delay in player turnover. Third, when payment to the departing player is not allowed, the outside option of a player has a nonmonotone impact on the payoff of the other player. Finally, in principal-agent relationships with limited liability, the profit maximizing relational contract never maximizes aggregate payoff if replacement of agent is not allowed, but allowing agent replacement restores the value maximization principle for a wide range of parameter values.

1 Introduction

Turnover of firm ownership and employment relations are prevalent. In labor economics, turnover is often viewed as a consequence of mismatch between workers and firms or redundancy of workers resulting from change in economic environment. More specifically, turnover is a consequence of "variation in the quality of the worker-employer match" (Jovanovic, 1979, p. 972). Such view is also expressed in Lazear (1998, p. 169), "If new workers and old workers were perfect substitutes for one another, there would be little reason to cycle workers into and out of the firm". The theories of entry and exit in the industrial organization literature share a similar spirit. Firms enter markets with imperfect knowledge of their profitabilities and exit when they learn that they are unfit to compete in the markets (e.g., Jovanovic, 1982). According to these theories, there would be no reason to replace an existing player unless the new player is believed to be of higher ability or better match.

In this paper, we investigate an underexplored benefit of player turnover in long-term relationships. It is a widely accepted wisdom that in repeated games with imperfect monitoring, some surplus has to be destroyed in equilibrium or otherwise it is impossible to support cooperative plays (Green and Porter, 1984). We analyze how such surplus destruction can be naturally mitigated through player turnover when the number of potential players of the repeated game exceeds the number of players required to play the repeated game. Our theory is built on the simple idea that when player(s) are suspected to have deviated from the equilibrium strategies, they will have to be punished if they continue to play the game. However, since new players are not responsible for the bad outcome, they can be spared of the otherwise necessary punishment. Therefore, by allowing player turnover following bad outcomes, the efficiency of the long-term relationship can be preserved in spite of imperfect monitoring. Summing up, player turnover allows the suspected deviator to be punished without punishing the relationship and other players in the game. This remains true even if the suspected deviators must be adequately compensated in order for them to leave voluntarily.

According to our theory, following a bad outcome, Pareto improvement can be created between the suspected deviator and the potential replacement due to the fact that the suspected deviator must be punished if she stays in the game but the incoming player does not have to be punished. And this gives rise to the incentive to trade the right to play the repeated game. Our theory predicts that turnover of players following a bad outcome creates Pareto improvement even if the new player is an identical replica of the existing player.[1] This implication distinguishes our theory from existing theories of turnover according to which turnover is a consequence of mismatch. Casual observations suggest that when the performance of a company improves following the change in management or ownership, the new manager or owner receives the credit for the improved performance and the improvement is attributed to the better skill or better decisions of new manager or new owner. One insight of our theory is that the improvement in performance of a relationship following player turnover may not be attributable to difference in player abilities. New players perform better simply because as part of the equilibrium of a bigger game, the punishment designed for the departing players does not have to be imposed on the new players.

In this paper, we explicitly distinguish the payoffs of the games from the payoffs of players and we characterize both sets of equilibrium payoffs. Such distinction is useful because often time the former payoffs are important to agents who do not have an active role in the repeated game but care about the efficiency of equilibrium of the game. For example, small share holders of public companies benefit from the cooperation in the repeated game played between the management and workers. Citizens of neighboring nations benefit from the cooperation in the repeated game played between leaders of these nations.

In the general setup of the paper, we develop a model building on the standard models of repeated game: time is discrete, and an abstract stage game will be played repeatedly in each period, possibly by different players. The novelty here is that, at the end of each period, all players (both those on the game and off the game) are required to make an entry/exit announcement. According to the announcements, a (prescribed) player transition function will determine who will play the game next period. In addition, a prescribed player transfer function will determine the transfers across players.

We show that, if the transition function and transfer function satisfy some technical conditions, then the game can be analyzed recursively. In particular, standard techniques like Abreu, Pierce, and Stacchetti (1990) (APS hereafter) on repeated games can be applied here to characterize the set of Perfect Public Equilibrium (PPE).

[1] In fact, turnover still creates Pareto improvement even if turnover is costly.

Moreover, if the outside options of the players (when they leave the game) is independent of the strategy of other players (this is satisfied, for example, if a player leaves the game, he can never return), then there is a more compact recursive formulation that greatly simplifies the analysis.

To illustrate the method developed in the general game, and also to highlight the economic consequences of replacement, we consider three applications.

In the first application, we look at the case when the stage game in the Prisoner's Dilemma. One feature of our theory of voluntary turnover is that, when payment to the departing player is allowed, the bargaining power of the departing player over the incoming player crucially influences the room for Pareto improvement and how the optimal relational contract with turnover is structured. The intuition is as follows. The Pareto improvement from replacing a suspected deviator and a new player arises from the difference in the payoffs of these players playing the continuation subgame. When the incumbent player has strong bargaining power over the potential replacement, she can capture a large portion of the Pareto improvement from the trade of the right to play the continuation subgame. In other words, it become more difficult to punish her deviation. Anticipating that, the incumbent player may lose the incentive to cooperate.

In the extreme case that the incumbent player has 100% of bargaining power over her replacement, no Pareto improvement can be created by turnover. However, as long as the incumbent players do not have 100% bargaining power over the new players, some Pareto improvement can be created. This is done through a second best arrangement requiring the incumbent player to stay in the "sour" relationship and receiving a low payoff for some periods before she is allowed to sell the right to the game to a new player. The delay of turnover will lead to some inefficiency and the duration of inefficient outcomes prior to replacement of players increases in the incumbent players' discount factors. In the end, we prove a folk theorem according to which, so long as the incumbent player does not have 100% bargaining power, as players become sufficiently patient, such punishment can be avoided and the entire feasible set of payoffs is sustainable as a sequential equilibrium.

Our second application is on principal-agent relationship with limited liability, using the setup in Levin (2003). Without limited liability, Levin (2003) characterizes the optimal relational contract and shows that the optimal relational contract can be implemented by stationary contracts. When there is a lower bound on the compensation the principal can offer to the agent, Fong and Li (2008) shows that the optimal relational contract (the one that maximizes the principal's payoff) is never efficient. Here, we characterize the optimal relational contract with limited liability, and we show that it has the following properties.

First, the optimal contract is efficient. Second, the structure of the relationship goes as follows. When a new relationship starts, the agent is paid the minimum wage w. The agent is fired if his continuation payoff falls below a low threshold. The agent receives permanent employment when his continuation payoff exceeds a high threshold. In this case, the optimal contract can be sustained by a stationary contract. Third, the expected payoff of the agent is nondecreasing over time. Fourth, the expected payoff of the principal is highest at the onset of the relationship and is nonincreasing over time in an employment relationship. Fifth, the principal's payoff is higher than that in a game without replacement, and the agent's payoff is lower than that in a game without replacement.

The results in this application reinforce the idea that replacement can mitigate the inefficiency in repeated game with public monitoring. More interestingly, note that while ending a relationship is a bad news for the principal in the game without replacement, it is a good news here: the expected payoff of the principal is highest at the onset of a relationship. Perhaps this helps explain why many firms find their "seasoned workers" too expensive. But any attempt to fire these workers will only backfire, as, Philip Schoonover, the CEO of Circuit City found out in a hard way, because it destroys the whole incentive system and leads to low morale.

Our final application also considers a repeated principal agent relationship, but no transfer is allowed. In addition, we assume that the principal cannot fire the agent. This describes, for example, the incentive problem facing individuals holding public offices in which monetary award is often shunned and firing can be costly. This example serves two purposes. First, we show that the outside option of the agent can affect the efficiency of the relationship in a non-monotonic way. We characterize the optimal relational contract without transfer and show that, if the outside option of the agent is smaller than his maxmin payoff in the game, then first best (effort) cannot be obtained. As the agent's outside option improves, there's a range in which first best is obtainable. And as agent's outside option improves further, the efficiency of the relationship starts to decrease. The analysis calls into attention into the recruiting policy: when firing is difficult, the ideal agent is one whose outside option is not too high or too low.

The second purpose of this application is to illustrate the subtle effect of entry and exit rule on the performance of a relationship. In particular, we show that if the agent is allowed to reenter the game, then the first best is possible even if the agent's immediate outside option is lower than his maxmin payoff in the game. This is because the possibility of reentry increases the de facto outside option of the agent. More interestingly, for the first best to be obtained, we need the re-entry to be neither too easy nor too hard. This is because if the re-entry is too easy, then the outside option can be too high as to damage the incentive of the agent inside the game.

For the rest of the paper, Section 2 reviews the related literature. Section 3 provides the general setup and two recursive formulations. Section 4-6 analyze three applications of the general model. Section 7 concludes.

2 Related Literature

In repeated games, the possibility that players can exit a relationship and form a new one is often viewed as a threat to sustainability of relationship. Several studies have focused on how to restore the sustainability of relationship in which replacement is not allowed. See, for example, Kandori (1992), Ghosh and Ray (1996), and Kranton (1996).

Demsetz (1988) informally discusses the view that "a new broom sweeps clean". Demsetz views takeover as an opportunistic behavior which allows the new management to break promises made by the previous management. Like other papers mentioned in the previous paragraph, player turnover is considered as a threat to sustainability of long-term relationships. In contrast, we model both player turnover and the release of the new player from the previous player's punishment as part of an efficient equilibrium instead of any player's opportunistic behavior and shows that such arrangement not only does not necessarily threaten the sustainability of relationship, but also can enhance its efficiency.

Our theory of turnover is also related the literature studying the trading of firm reputations pioneered by Tadelis (1999, 2002, 2003) and Mailath and Samuelson

(2001). According to their theories, when a firm's reputation is summarized by its past performances, new entrants can buy the good reputation firms which have to exit for exogenous reasons, giving rise to value of company names. One limitation of these theories is that trademarks have values only when their transactions are unobservable by consumers (Deb, 2007). Our theory of turnover is different from these theories in many ways. Most notably, players in our theory do not have reputations or types. It is easy to adapt our theory to explain turnover of firm ownership and our theory predicts transaction of trademark to take place following bad outcome, performance to improve following ownership change, and such transaction will take place only they are observable. The stark contrasts in predictions suggest our theory as a complement to theirs.

Our analysis of agent turnover in the principal-agent relationship with limited liability is related to MacLeod and Malcomson (1989, 1993, 1998). In their studies, turnover of workers enforced by the employer threatens the sustainability of relational incentive contracts. We make different assumption on observability in our analysis, so agent turnover in our model not only does not diminish the sustainability of the relational contract, it actually increases the principal's payoff.

3 Setup

3.1 General Setup

Time is discrete and infinite: $t \in \{1, ..., \infty\}$. The discount factor across periods is given by $\delta \in (0,1)$. Consider a game (to be described below) that's being played repeatedly each period, possibly by different players.

3.1.1 Players

Denote the set of players as $I \subset R$. In period t, denote the set of players inside the game as an *ordered* set

$$I_t^{In} = \{i_t^1, ..., i_t^n\}.$$

The set of such collections is denoted by $\mathbf{I}^{In}$. We use i_t^k as the player who's in the k^{th} position of the game in period t.

Define the set of players outside the game in period t as

$$I_t^{Out} = I \setminus I_t^{In}.$$

The set of such collections is denoted by $\mathbf{I}^{Out}$.

3.1.2 Actions and Outcomes

For each player, there are two types of actions: actions in the stage game, and actions in deciding the entry and exit of the game.

For player $i \in I$ in period t, his stage game action $a_t^i \in A_t^i$. We assume that

$$A_t^i = \begin{cases} A_k & \text{if } i = i_t^k \text{ for some } k \in \{1, ..., n\}; \\ \{\varnothing\} & \text{otherwise.} \end{cases}$$

Let $A = \times_{k=1}^n A_k$. Each action profile $a \in A$ induces a signal $y \in Y$ with generalized density $f(y|a)$.

Player $i's$ entry and exit action $x_t^i \in X_t^i$. We assume that

$$X_t^i = \begin{cases} (\{0,1\} \times R)^I & \text{if } i = i_t^k \text{ for some } k \in \{1, ..., n\}; \\ \{0,1\} \times R & \text{otherwise.} \end{cases}$$

For $i \in I_t^{In}$, we use 0 to denote the intension to "stay" and 1 to denote the intention to "exit". For $i \in I_t^{Out}$, we use 0 to denote intention to "not enter" and 1 to denote the intention to "enter". When players state their intentions, they also announce a message that determines the amount(s) of transfer according to a transfer function to be described below. Finally, let $X_t = \times_{i=1}^I X_t^i$.

3.1.3 Stage Payoffs

The stage game payoffs of the players inside the game are given by a continuous function $u : A \times Y \to R^n$. When the player is outside the game, his payoff is given by $\underline{u}^i$. In other words,

$$u_t^i = \begin{cases} u_k(a_t, y) & \text{if } i = i_t^k \text{ for some } k \in \{1, ..., n\}; \\ \underline{u}^i & \text{otherwise.} \end{cases}$$

There is no costs associated with entry and exit decisions. These will be described in the transfer function subsection.

3.1.4 Transition Function of Players

In period t, let N_t be a function that maps the player list in period $t-1$, I_{t-1}^{In}, and the associated entry and exit action profile x_{t-1}, into a player list in period t:

$$N_t : \mathbf{I}^{In} \times X_t \rightarrow \mathbf{I}^{In}.$$

For the general game, it is possible that this transition function depends on the time periods. In our examples and analysis below, we will focus on transition functions that are time invariant. The stationarity of the transition function with respect to time makes it possible for a recursive representation of the game.

3.1.5 Public History and Public Strategy

Define each public history at the beginning of period t as

$$h_t = \{I_1^{In}, y_1, x_1, ..., y_{t-1}, x_{t-1}, I_t^{In}\},$$

and H_t be the set of such histories. In particular, $h_1 = \{I_1^{In}\}$. Compared to a standard setup without replacement, we also include the indentities of the players in the game (I_t^{In}) and the decisions of entry and exit of the game (x_t) to be part of the public history.

A public strategy of player i is a sequence of functions $\{s_t^i\}_{t=1}^{\infty}$ such that for each time t,

$$s_t^i : H_t \times H_t \cup \{y_t\} \rightarrow A_i \times X_i.$$

In addition to choosing the stage game action, as in standard repeated games without replacement, the public strategies here also specify an entry/exit decision after the stage game outcome is realized.

We use σ_t^i as the mixed public behavioral strategy in period t and σ^i as the mixed strategy of player i.

3.1.6 Transfer Function

At the beginning of period $t > 1$, we denote the transfer function (associated with the entry and exit decisions of the players) as

$$T_t : H_t \times \Sigma^I \rightarrow R^I,$$

where Σ^I is the set of mixed strategies by players. We intentionally leave the transfer function general to accomodate a wide range of applications. The transfer function will take a simple form in the applications we analyze in Sections 4-6.

3.1.7 Payoffs

If each player $i \in I$ follows strategy s^i, then the payoff vector of the I players are given by

$$(1-\delta)\sum_{t=1}^{\infty}\sum_{h_t \in H_t} \delta^{t-1} \Pr(h_t|s)(f(y_t|s_t^a(h_t))(U_t(s_t^a(h_t), y_t|h_t) + \delta T_{t+1}(h_{t+1}, s)),$$

where $s_t^a(h_t)$ is the stage game action profile induced in period t by strategy s following public history h_t, and $U_t^i = u_k$ if $i = i_t^k$ for some k and $U_t^i = \underline{u}^i$ otherwise. When mixed strategies are played, the payoffs can be defined in the standard fashion, but the notations will be more complicated so we omit it here.

Following each history h_t, we denote the continuation payoffs of the players at the beginning of period t as $V(h_t, \sigma) \in R^I$. To abuse the notation somewhat, we can also define the continuation payoff of the player following the t^{th} period payoff as $V(h_t, y_t, \sigma)$.

3.1.8 Equilibrium Concepts

We look for the Perfect Public Equilibrium (PPE) of the game. In particular, we require that for each public history h_t and $h_t \cup y_t$, we have

$$\begin{aligned} \sigma^i|h_t &\in \arg\max_{\widetilde{\sigma^i}} V^i(h_t^i, \widetilde{\sigma^i}, \sigma^{-i}) \\ \sigma^i|h_t, y_t &\in \arg\max_{\widetilde{\sigma^i}} V^i(h_t^i, y_t, \widetilde{\sigma^i}, \sigma^{-i}) \end{aligned}$$

Since this is a game with public monitoring that's continuous at infinity, we can redefine the PPE using one-stage deviation principle. In particular, we will require that, at the beginning of period t for each history h_t, we require that a_t^i be a best response for player i (holding the strategies of $-i$ and i's in future periods. At the end of end of period t following h_t, y_t, we have that x_t^i is a best response.

3.2 Examples

In this subsection, we provide three examples of games involving player turnover. Some version of each of these examples will be analyzed formally in Sections 4-6.

3.2.1 Partnership/Managers

In this game, the set of players are $I = \{1, 2, ..., 2M\}$, where again it is possible that M is infinity. In each period, there are exactly two partners in the game. In period 1, $I_1^{In} = \{1, 2\}$.

In the stage game, each partner chooses to put in effort or not. In other words, $A_t^1 = A_1 = A_t^2 = A_2 = \{E, S\}$.

In the stage game, there are two possible outcomes: $Y = \{y_L, y_H\}$. If the both partners put in effort, then y_H occurs with probability p. If only one partner puts in effort, then y_H occurs with probability $q < p$. If neither puts in effort, then then y_H occurs with probability $r < q$. The normalized payoff of the game is captured by the following matrix:

	E	S
E	(y, y)	$(0, z)$
S	$(z, 0)$	(x, x)

Again, we assume that $2y > z > y > x > 0$.

At the end of each period, any of the two partners may leave the market. If this happens, his normalized payoff in all future periods is $\underline{u} < x$.

There are two possible ways that the agent leaves the game. We assume that if one partner (both partners) leaves, then the next player(s) on the list will enter (assuming that entry is a dominant strategy, which we will comment on later).

Suppose player i and j are on the game in period t, and the next players on the list are k and $k+1$, then the transition function satisfies

$$\begin{aligned} N_t(\{i,j\},x_t) &= \{i,j\} && \text{if } x_t^i = 1 \text{ and } x_t^j = 1 \\ &= \{i,k\} && \text{if } x_t^i = 1 \text{ and } x_t^j = 0 \\ &= \{j,k\} && \text{if } x_t^i = 0 \text{ and } x_t^j = 1 \\ &= \{k,k+1\} && \text{if } x_t^i = 0 \text{ and } x_t^j = 0 \end{aligned}$$

Note again that this transition function is stationary, given the availability of the players.

Finally, if at least one partner stays, then we assume that there's no transfer. Otherwise, the exiting partners will Nash Bargain with the entering partners. More formally, suppose $I_t^{In} = \{i,j\}$, the current history is $h_t \cup y_t$, and the next players on the list are k and $k+1$, then

$$T_{t+1}(h_{t+1},\sigma) = 0 \quad \text{for } I_{t+1}^{In} \neq \{k,k+1\}$$

Otherwise,

$$T_{t+1}^i(h_{t+1},\sigma) = \beta(V^k(h_t,y_t,\{k,k+1\},\sigma) - V^i(h_t,y_t,\{i,j\},\sigma);$$

$$\begin{aligned} T_{t+1}^j(h_{t+1},\sigma) &= \beta(V^{k+1}(h_t,y_t,\{k,k+1\},\sigma) - V^j(h_t,y_t,\{i,j\},\sigma); \\ T_{t+1}^k(h_{t+1},\sigma) &= -\beta(V^k(h_t,y_t,\{k,k+1\},\sigma) - V^i(h_t,y_t,\{i,j\},\sigma); \\ T_{t+1}^{k+1}(h_{t+1},\sigma) &= -\beta(V^{k+1}(h_t,y_t,\{k,k+1\},\sigma) - V^j(h_t,y_t,\{i,j\},\sigma). \end{aligned}$$

Notice that the transfer function is again stationary here, and it does not depend on the strategy of the players conditional on the players in the game.

Comment: we have not really considered the IC of entering players. And for this matter, what happens if the current players leave and no new players come in. There are three modelling choices. The first is that if some players leave and not enough new ones come in, the game ends forever. The second is that the game stays idle until there are new players come in and fill the gap. The third is that, if not enough new ones come in, the existing ones cannot leave.

3.2.2 Principal Agent with Limited Liability

In this game, the set of players are $I = \{1, 2, ..., M\}$, where again it is possible that M is infinity. Let player 1 be the principal, and the rest of players are the set of potential agents. In each period, there is exactly one agent inside the game. In period 1, $I_1^{In} = \{1, 2\}$.

At the beginning of each period t, the principal offers the agent $w_t \geq \underline{w}$. Therefore, $A_t^1 = A_1 = [\underline{u}, \infty)$. The agent chooses effort $e_t \in \{0, 1\}$, so $A_t^i = A_2 = \{0, 1\}$ for $i \geq 2$. If the agent puts in effort, the outcome Y is y with probability p and 0 with probability $1 - p$. When no effort is put in, the outcome is y with probability $q < p$. The effort costs c. We assume that

$$py - c > \underline{u} + \underline{v} \geq \underline{v} > qy,$$

where $\underline{v}$ is the principal's outside option.

At the end of each period, if the principal choose the leave the game, the game ends forever: the principal receives $\underline{v}$ and all agents receive $\underline{u}$ per period in all future periods. At the end of each period, the agent may also leave the game. If this happens, his normalized payoff in all future periods is $\underline{u} < x$.

There are two possible ways that the agent leaves the game. First, the agent may choose to leave. Second, the principal can fire the agent. If either happens, the next agent in the player list enters the game. More formally, the player transition function satisfies

$$\begin{aligned} N_t(\{1, i), x_t) &= (1, i + (1 - 1_{\{x_t^i = 1, x_t^1 = 1\}})) \quad \text{if } i < M \\ &= (1, 2 + (M - 2)1_{\{x_t^i = 1, x_t^1 = 1\}}) \quad \text{if } i = M \end{aligned}$$

Note again that the transition function is stationary. Moreover, when the player leaves, his future payoff is independent of the strategies of other players.

Finally, if the agent stays or quits, then there's no transfer. If the agent is fired, we assume that the principal may pay a severance pay S_t to the agent. Suppose

$I_t^{In} = \{1, i\}$, then

$$T_{t+1}(h_{t+1}, \sigma) = 0 \quad \text{for } I_{t+1}^{In} = \{1, i\} \text{ or } x_t^1 = 1.$$

Otherwise,

$$\begin{aligned} T_{t+1}^1(h_{t+1}, \sigma) &= -S_t; \\ T_{t+1}^i(h_{t+1}, \sigma) &= S_t; \\ T_{t+1}^j(h_{t+1}, \sigma) &= 0 \quad \text{for } j \neq 1, i. \end{aligned}$$

Notice that the transfer function is again stationary here, and it does not depend on the strategy of the players conditional on the players in the game.

3.2.3 Politicians and Citizen

In this game, the set of players are $I = \{1, 2, ..., M\}$, where it is possible that M is infinity. Let player 1 be the citizen, and the rest of players are politicians. In each period, there is exactly one politician inside the game. In period 1, $I_1^{In} = \{1, 2\}$.

In the stage game, the citizen can choose to trust or distrust the politician. In other words, $A_t^1 = A_1 = \{T, NT\}$. The politician can choose an action that favors private benefit b or the public interest B. In other words, $A_t^2 = A_2 = \{b, B\}$.

If the citizen doesn't trust the politician, then both sides receive $x > 0$. If he trusts the politician, then there are two possible outcomes, $Y = \{y_L, y_H\}$. If the politician chooses action B, then y_H occurs with probability p. If the politician chooses action b, then y_H occurs with probability $q < p$. The normalized payoff of the game is captured by the following matrix:

	T	NT
B	(y, y)	(x, x)
b	$(z, 0)$	(x, x)

We assume that $2y > z > y > x$.

At the end of each period, the politician may leave. If he leaves, he receives a normalized payoff of $\underline{u}$, which may or may not be smaller than x in all future periods. We assume that $\underline{u} < y$.

For the player transition function, we assume that if the politician chooses to stay for the next period, he stays. Otherwise, the next politician in the player list takes the position. More formally,

$$\begin{aligned} N_t(\{1,i),x_t) &= (1, i+(1-x_t^i)) \quad \text{if } i<M \\ &= (1, 2+(M-2)x_t^i) \;\; \text{if } i=M \end{aligned}$$

Finally, we assume that there will be in transfers in involved in the transitional process, i.e., $T(.)=0$.

3.3 Recursive Structures

3.3.1 General Case

In this subsection, we study how the problem can be formulated in a recursive structure. The key conditions required is that the transfer function and the transition function need to be stationary (do not depend on the time period.) Moreover, we need the transfer function at the beginning of period $t+1$ to depend on four elements only: 1) the identity of the players in period t, 2) the identity of the players in period $t+1$, 3) the continuation payoffs of all the players (at the beginning of $t+1$) if those in period t stay on in $t+1$, and 4) the continuation payoffs of all the players (at the beginning of $t+1$). More formally, for all t, h_t, σ, we have

$$T_{t+1}(h_{t+1}, y_t, \sigma) = T(I_t^{In}, I_{t+1}^{In}, V(h_t, y_t, I_t^{In}, \sigma), V(h_t, y_t, I_{t+1}^{In}, \sigma)).$$

Note that this condition is satisfied in all of the examples above.

With these conditions, we can adapt the standard techniques by Abreu, Pierce, and Stacchetti (1990) (APS) to analyze the game. In particular, the state variable will be the set of payoffs and the set of players who are in the game. Take $V^G \subset F^G \subset R^I$, where F^G is the convex hull of the set of feasible payoffs,

Definition 1 *An action pair (a, x) and player set I^{In} are enforceable wrt to $V^G \times \{I^{New}\}$ if there exists two value functions Γ_{In}^G and Γ_{New}^G from Y to V^G and a set of* *players $I^{New}(y) \in \{I^{New}\}$ such that for each player $i \in I$, we have*

$$\begin{aligned}
(a^i, x^i(y)) &\in \arg\max_{(\widetilde{a^i}, \widetilde{x^i}(y))} \int [(1-\delta)(u(\widetilde{a^i}, a^{-i}, y) + \\
&\quad \delta(T(I^{In}, N(I^{In}, (\widetilde{x^i}(y), x^{-i}(y))), \Gamma^G_{In}(y), \Gamma^G_{New}(y)) + \Gamma^G_{New}(y))] f(y|\widetilde{a^i}, a^{-i}) dy \\
I^{New}(y) &= N(I^{In}, x(y))
\end{aligned}$$

We define $B^G((V^G, \{I^{New}\}))$ as the set of payoffs and the associated players such that there exists an action pair (a, x) and player set I^{In} enforceable wrt to $V^G \times \{I^{New}\}$ with the associated Γ^G_{In}, Γ^G_{New} and set of players $I^{New}(y)$ such that

$$\begin{aligned}
&B^G((V^G, \{I^{New}\})) = \{(v^G, I^{In}), \text{ such that} \\
&v^G = \int f(y|a)[(1-\delta)u(a,y) + \delta(T(I^{In}, N(I^{In}, x)), \Gamma^G_{In}(y), \Gamma^G_{New}(y)) + \Gamma^G_{New}(y))dy \\
&\qquad I^{New}(y) = N(I^{In}, x(y)) \\
&\qquad (\Gamma^G_{New}(y), I^{New}(y)) \in V^G \times \{I^{New}\}\}
\end{aligned}$$

The formulation above gives a recursive structure to the problem, with the state variable being the cross product of the continuation payoffs and the set of players on the game, so that techniques developed by APS can be applied here. For example, we can define a set $(V^G, \{I^{New}\})$ to be self-generating if

$$(V^G, \{I^{New}\}) \subset B^G((V^G, \{I^{New}\})).$$

It is then clear from APS that if a set is self-generating, then its first component contains the set of PPE payoffs.

The dimension of this recursive structure is $|I|$. When the number of potential players are indeed small, this representation becomes a very good one because it captures rich strategic dynamics both inside and outside the game. For example, in the politician/citizen case, this representation allows for the possibility that a politician may re-enter the game. It then follows that the outside option of a politician will depend on the expected time of re-entry. In such cases, multiple equilibrium can arise: one would like to stay in the position longer if one expects the others will stay in the position longer.

However, when the number of potential player is large, the representation in the previous section can be unwieldy. In many applications (most of them in this paper) we do not require such a general representation. Therefore, instead of deriving properties of PPE payoffs with this formulation, we derive these properties in a compact case in the following subsection and we provide an example in Section 6 to illustrate how the general representation might be used.

3.3.2 Simplified Representation

In this subsection, we describe a simplified case. In this case, under some conditions there exists a simpler recursive representation that has dimension n, the number of players on the game.

The key conditions for this representation are that, once some players leaves, their continuation payoffs outside the game (net of the transfer function) are independent of the strategies of other players. This condition is satisfied say, if once the players leave the game, they never return to the game and always receive their outside options. Of course, this is a good assumption only when the number of potential players is large. In some cases, there are cases this recursive representation misses out interesting cases.

In particular, let $I = \{1, ..., \infty\}$. $I_1^{In} = \{1, ..., n\}$. We assume that there exists a set $M(t)$ strictly increasing in t such that

$$N_t(I_{t-1}^{In}, x_t) \subset M(t) \cup I_{t-1}^{In}.$$

This assumption makes sure that, if a player ever exits, he receives $\underline{u}$ in all future periods. Moreover, if a player $i \in M(t)$ had a chance to join the game in period t (through some outcome x_t) but he didn't, he won't have the chance to join the game in all future periods.

We also assume that there is no transfer to players who are not in the game and are not about to enter the game.

$$T^i(I^{In}, N(I^{In}, x), v_{In}, v_{new}) = 0 \quad \text{for } i \notin I^{In} \cup N(I^{In}, x).$$

In other words, transfer takes place only among the "active players": those staying on, those exiting, and those entering.

With these assumptions, we are ready to define the following definition of enforceability.

Take $W \subset R^n$, then an mixed action profile $(\alpha, x(y))$ is enforceable with respect to W if there exists two functions $\Gamma_{In} : Y \to W$, $\Gamma_{New} : Y \to W$, such that for all $i \in I^{In}$,

$$\begin{aligned}(\alpha^i, x^i(y)) \;\in\; & \arg\max_{(\widetilde{\alpha^i}, \widetilde{x^i}(y))} \int [(1-\delta)(u^i(\widetilde{\alpha^i}, \alpha^{-i}, y) + T^i(I^{In}, N(I^{In}, (\widetilde{x^i}(y), x^{-i}(y))), \Gamma_{In}(y), \Gamma_{New}(y))) \\ & \delta(\underline{u}^i + 1\{i = i^k \in N(I^{In}, (\widetilde{x^i}(y), x^{-i}(y)))\}(\Gamma^k_{New}(y) - \underline{u}^i))] f(y|\widetilde{\alpha^i}, \alpha^{-i}) dy.\end{aligned}$$

Now take $W \subset R^n$, we define

$$\begin{aligned}B(W) &= \{w : w^i = \int [(1-\delta)(u^i(a, y) + T^i(I^{In}, N(I^{In}, x(y)), \Gamma_{In}(y), \Gamma_{New}(y))) \\ +\delta(\underline{u}^i + 1\{i &= i^k \in N(I^{In}, (\widetilde{x^i}(y), x^{-i}(y)))\}(\Gamma^k_{New}(y)) - \underline{u}^i)] f(y|\widetilde{a^i}, a^{-i}) dy, \\ & \quad \text{where } (a, x(y)) \text{ is enforceable wrt } W \text{ with the associated } \Gamma_{In}, \Gamma_{New}\}\end{aligned}$$

With this definition, we can carry out our analysis just as in APS. In particular, we list the following properties of the B operator and the PPE payoff set.

Property 1: If $W \subset B(W)$, then each $w \in W$ is a PPE payoff.

Property 2: The set of PPE payoff E is the largest set in R^n such that

$$E = B(E).$$

Property 3: E is compact and convex.

Property 4: B is monotone:

$$B(W_1) \subset B(W_2) \quad \text{if } W_1 \subset W_2.$$

The standard procedure of finding E is to use the property that B is a monotone operator, so that

$$E = \lim_{n \to \infty} B^n(F),$$

where F is the set of feasible payoffs. As we will see in Section X, there are some special cases where we can find E directly by exploiting the recursive structure.

4 Application 1: Player Turnover in the Prisoner's Dilemma

In this section, we analyze the special case in which the stage game is the Prisoner's Dilemma with the expected instantaneous payoffs from the action profiles:

	E	S
E	(y, y)	$(0, z)$
S	$(z, 0)$	(x, x)

The underlying structure that gives rise to such payoffs is as follows.[2] Let $a_i \in \{E, S\}$ denote the action of player $i \in \{1, 2\}$ and $a = (a_1, a_2)$ denote the action profile. Let the cost of action E be $c > 0$ and the cost of action S be 0. The realized individual output is $y \in \{\underline{y}, \overline{y}\}$, where $\underline{y} < \overline{y}$. Suppose

$$\rho(\overline{y}|a) = \begin{cases} p, & \text{if } a = EE \\ q, & \text{if } a = SE \text{ or } ES \\ r, & \text{if } a = SS. \end{cases}$$

Further assume that

$$\begin{aligned} p\overline{y} + (1-p)\,\underline{y} - c &= y, \\ q\overline{y} + (1-q)\,\underline{y} &= z, \\ q\overline{y} + (1-q)\,\underline{y} - c &= 0, \\ r\overline{y} + (1-r)\,\underline{y} &= x. \end{aligned}$$

[2] This example borrows heavily from Mailath and Samuelson (2006) except that we allow for player turnover.

This gives rise to the expected instantaneous payoff specified in this section.

In this section, we restrict to strongly symmetric equilibrium. In other words, we only focus on equilibria in which the action profile that follows any public history is symmetric: $\sigma_1(h^t) = \sigma_2(h^t)$ for all public histories h^t.

4.1 Repeated Prisoner's Dilemma without Player Turnover

Suppose upon seeing a bad outcome, the players assign probability ϕ of reverting to defecting perpetually. Then

$$V = (1-\delta)\,y + \delta\left[pV + (1-p)\left(\phi V + (1-\phi)\,x\right)\right].$$

This can be rewritten as

$$V = \frac{(1-\delta)\,y + \delta\,(1-p)\,(1-\phi)\,x}{1-\delta\,(p+(1-p)\,\phi)}. \tag{1}$$

The Pareto dominant equilibrium is achieved by solving

$$\max_{\phi} V$$

subject to

$$V \geq (1-\delta)\,z + \delta\left[qV + (1-q)\left(\phi V + (1-\phi)\,x\right)\right].$$

The constraint can be rewritten as

$$V \geq \frac{(1-\delta)\,z + \delta\,(1-q)\,(1-\phi)\,x}{1-\delta\,(q+(1-q)\,\phi)}. \tag{2}$$

Lemma 4.1 *When turnover of players is not allowed in the infinitely repeated Prisoner's dilemma, the set of strongly symmetric sequential equilibrium payoffs for each individual player is*

$$\left[x, \frac{(1-q)\,y-(1-p)\,z}{(p-q)}\right]$$

when $\delta \in [\hat{\delta}, 1)$, *where* $\hat{\delta} = (z-y)\,/\,[p\,(z-x) - q\,(y-x)]$ *and is* $\{x\}$ *when* $\delta \in (0, \hat{\delta})$.

Notice that the upper bound on the individual payoff $[(1-q)\,y-(1-p)\,z]\,/\,(p-q)$ is strictly less than y and is independent of δ. This immediately implies that the Folk Theorem fails.

Proof. It follows from (1) and (2) that players have no incentive to play S if and only if

$$\frac{(1-\delta)\,y+\delta\,(1-p)\,(1-\phi)\,x}{1-\delta\,(p+(1-p)\,\phi)} \geq \frac{(1-\delta)\,z+\delta\,(1-q)\,(1-\phi)\,x}{1-\delta\,(q+(1-q)\,\phi)}$$
$$\Leftrightarrow \phi \leq \frac{\delta\,(p\,(z-x)-q\,(y-x))-(z-y)}{\delta\,(p\,(z-x)-q\,(y-x)-(z-y))} \equiv \bar{\phi}.$$

Since V increases in ϕ, optimality requires that $\phi=\bar{\phi}$. This implies that the maximum V attainable is

$$\begin{aligned} V &= \frac{(1-\delta)\,y+\delta\,(1-p)\,(1-\bar{\phi})x}{1-\delta(p+(1-p)\,\bar{\phi})} \\ &= \frac{(1-q)\,y-(1-p)\,z}{(p-q)}. \end{aligned}$$

The existence some $\phi\in[0,1]$ such that $\phi\leq\bar{\phi}$ requires that $\bar{\phi}\geq 0$, i.e.,

$$\delta \geq \frac{z-y}{p\,(z-x)-q\,(y-x)} \equiv \bar{\delta}.$$

$$\frac{d}{d\delta}\bar{\phi} = \frac{-\,(z-y)}{(z-y+q\,(y-x)-p\,(z-x))} < 0$$

■

4.2 Repeated Prisoner's Dilemma with Player Turnover

Assume that there are infinitely many identical agents who are potential players of one PD game.

Now, we define the *right to the game* that is to be transferred when replacement of players takes place. The right to the game consists of the right to play the Prisoner's Dilemma and the right to sell the right to the game to a potential player.

Definition 2 (Equilibrium) *There are two states on the equilibrium path:* normal state *and* temporary punishment state*. The equilibrium starts out in the normal state. In the normal state of the game, both existing players play E. If* $y=\underline{y}$ *in a period, then a public randomization device will assign probability* $\phi\in[0,1]$ *of returning to the normal state and probability* $(1-\phi)$ *of entering the temporary punishment state.*

In the temporary punishment state, both players play S for $k \geq 0$ periods. At the end of the temporary punishment state, both players each sell the right to the game to a potential player. We assume that each existing player will Nash bargain with a randomly selected potential player over a transaction price for the right to the game.

If any observable deviation from the equilibrium path, including not selling the right to the game while supposed to and selling the right to the game at a time earlier than supposed to, is observed, the permanent punishment state *is triggered. The permanent punishment state is characterized by current and future players of the game perpetually playing SS. The permanent punishment state is an absorbing state and once in that state even turnover takes place in the future, new players will continue to play SS perpetually. Therefore, once the permanent punishment state ensues, $V_x = x$ and any (normalized) transaction price will be x.*[3]

Let V be the value of the incumbent player during the normal state. Let τ be the (normalized) transaction price. Each player receives an expected instantaneous payoff y. With probability p, the output is high and players return to the normal state. With probability $(1-p)$, the output is low and players assign probability ϕ of returning to the normal state and probability $(1-\phi)$ of entering the temporary punishment phase for k periods and then selling the right to the game at the price τ in the $(k+1)$th period. Therefore, the (normalized) expected value of an incumbent player is

$$V = (1-\delta)\, y + \delta \left[pV + (1-p)\left(\phi V + (1-\phi)\left((1-\delta)\left(1+\delta+...+\delta^{k-1}\right)x + \delta^k \tau\right)\right)\right]. \tag{3}$$

Suppose the incumbent player's bargaining power is β. Then the Nash bargained price will be

$$\tau = x + \beta\,(V - x)\,. \tag{4}$$

Plugging (4) into (3), the latter can be rewritten as

$$V = (1-\delta)\, y + \delta \left[pV + (1-p)\left(\phi V + (1-\phi)\left(x + \delta^k \beta\,(V-x)\right)\right)\right]. \tag{5}$$

After rearranging terms, the incumbent player's value during the normal state can be

[3] That means the actual price of transaction is $x/(1-\delta)$.

rewritten as

$$V = \frac{(1-\delta)\,y + \delta\,(1-p)\,(1-\hat{\phi}\,(\phi,\beta,k))x}{1-\delta(p+(1-p)\,\hat{\phi}\,(\phi,\beta,k))}, \tag{6}$$

where

$$\hat{\phi}\,(\phi,\beta,k) = \phi + \delta^k \beta\,(1-\phi)\,. \tag{7}$$

Players have no incentive to deviate from playing the equilibrium strategy during the normal state if

$$\begin{aligned} V &\geq (1-\delta)\,z + \delta\left(qV + (1-q)\left(\phi V + (1-\phi)\left(x + \delta^k \beta\,(V-x)\right)\right)\right) \\ &= (1-\delta)\,z + \delta\left(qV + (1-q)\left(\hat{\phi}\,(\phi,\beta,k)\,V + (1-\hat{\phi}\,(\phi,\beta,k))x\right)\right), \end{aligned}$$

which can be rewritten as

$$V \geq \frac{(1-\delta)\,z + \delta\,(1-q)\,(1-\hat{\phi}\,(\phi,\beta,k))x}{1-\delta(q+(1-q)\,\hat{\phi}\,(\phi,\beta,k))}. \tag{8}$$

Define U as the total payoff of all the players which ever play one of the role in the infinitely repeated Prisoner's Dilemma and call it the *role payoff* of the game. Note that the period payoff of the game is y except during the temporary punishment state it is x. Also note that following replacement of players, the normal state is restored. Therefore, the role payoff is

$$U = (1-\delta)\,y + \delta\left[pU + (1-p)\left(\phi U + (1-\phi)\left(x + \delta^k\,(U-x)\right)\right)\right],\ i = 1,2. \tag{9}$$

Note that (9) is essentially (5) with $\beta = 1$. Now, rewrite (9) as

$$U = \frac{(1-\delta)\,y + \delta\,(1-p)\,(1-\hat{\phi}\,(\phi,1,k))x}{1-\delta(p+(1-p)\,\hat{\phi}\,(\phi,1,k))}. \tag{10}$$

Proposition 4.1 *Let $\left[x,\overline{V}\right]$ be the equilibrium payoff sets of players and $\left[x,\overline{U}\right]$ be the equilibrium role payoff sets. When $\beta = 1$, turnover of players does not create Pareto improvement and $\overline{U} = \overline{V}$ for $\delta \in (0,1)$. Suppose $\beta \in (0,1)$. Then there exist*

$\tilde{\delta}(\beta) \in (\hat{\delta}, 1)$, where $\tilde{\delta}'(\beta) > 0$, such that *when* $\delta \in (0, \hat{\delta})$,

$$\overline{U} = \overline{V} = x;$$

when $\delta \in [\hat{\delta}, \tilde{\delta}(\beta))$,

$$\overline{V} = \frac{(1-q)\,y - (1-p)\,z}{(p-q)} < \overline{U} = \frac{(1-\delta)\,y + \delta\,(1-p)\,(1-\frac{\bar{\phi}}{\beta})x}{1-\delta(p+(1-p)\,\frac{\bar{\phi}}{\beta})} < y;$$

when $\delta \in [\tilde{\delta}(\beta), 1)$,

$$\overline{V} = \frac{(1-q)\,y - (1-p)\,z}{(p-q)} < \overline{U} = y$$

Proof. Plugging (6) into (8), we can rewrite the latter as

$$\frac{(1-\delta)\,y + \delta\,(1-p)\,(1-\hat{\phi}\,(\phi,\beta,k))x}{1-\delta(p+(1-p)\,\hat{\phi}\,(\phi,\beta,k))} \geq \frac{(1-\delta)\,z + \delta\,(1-q)\,(1-\hat{\phi}\,(\phi,\beta,k))x}{1-\delta(q+(1-q)\,\hat{\phi}\,(\phi,\beta,k))},$$
$$\Leftrightarrow \hat{\phi}\,(\phi,\beta,k) \leq \frac{z-y+\delta\,(q\,(y-x)-p\,(z-x))}{\delta\,(z-y+q\,(y-x)-p\,(z-x))} = \bar{\phi}.$$

If $\beta = 1$, then according to (6) and (10), $U = V$. In this case, finding the most efficient equilibrium is equivalent to solving

$$\max_{k,\phi} V$$
$$\text{subject to} \quad \phi + \delta^k\,(1-\phi) \leq \bar{\phi},$$

the solution to which is $\phi + \delta^k\,(1-\phi) = \bar{\phi}$. It is easy to verify that $\overline{U} = \overline{V} = [(1-q)\,y - (1-p)\,z]/\,(p-q)$ if $\delta \geq \hat{\delta}$ and $U = V = x$ otherwise.

Now suppose $\beta \in (0,1)$. Finding the most efficient equilibrium is equivalent to solving

$$\max_{k,\phi} U = \frac{(1-\delta)\,y + \delta\,(1-p)\,(1-\hat{\phi}\,(\phi,1,k))x}{1-\delta(p+(1-p)\,\hat{\phi}\,(\phi,1,k))}$$
$$\text{subject to} \qquad \phi + \delta^k\beta\,(1-\phi) \leq \bar{\phi}.$$

First, since U increases in $\hat{\phi}\,(\phi,1,k)$, the maximization problem can be simplified as

$$\max_{k,\phi} \hat{\phi}\,(\phi,1,k) = \phi + \delta^k\,(1-\phi)$$
$$\text{subject to} \qquad \phi + \delta^k\beta\,(1-\phi) \leq \bar{\phi}.$$

Since $\hat{\phi}(\phi, 1, k)$ increases in ϕ and δ^k, optimality requires that the incentive constraint holds in equality, i.e., $\phi + \delta^k \beta (1 - \phi) = \bar{\phi}$ or equivalently $\phi = (\bar{\phi} - \delta^k \beta) / \left(1 - \delta^k \beta\right)$. Therefore, maximization problem can be further simplified as

$$\max_k \left[\left(1 - \delta^k\right) \frac{(\bar{\phi} - \delta^k \beta)}{(1 - \delta^k \beta)} + \delta^k \right].$$

Since

$$\frac{d}{d\delta^k} \left[\left(1 - \delta^k\right) \frac{(\bar{\phi} - \delta^k \beta)}{(1 - \delta^k \beta)} + \delta^k \right] = \frac{(1 - \bar{\phi})(1 - \beta))}{\left(1 - \delta^k \beta\right)^2} > 0,$$

it is optimal to set $k = 0$ (i.e., maximize δ^k) if there exist $\phi \in [0, 1]$ such that $\phi + \delta^k \beta (1 - \phi) \leq \bar{\phi}$ can be satisfied, which is the case if

$$\beta \leq \bar{\phi} = \frac{z - y + \delta (q (y - x) - p (z - x))}{\delta (z - y + q (y - x) - p (z - x))}, \tag{11}$$

or equivalently

$$\begin{aligned} \delta &\geq \frac{z - y}{p (z - x) - q (y - x) - \beta (p (z - x) - q (y - x) - (z - y))} \\ &= \frac{\hat{\delta}}{1 - \beta \left(1 - \hat{\delta}\right)} \equiv \tilde{\delta}(\beta) > \hat{\delta}. \end{aligned}$$

Obviously,

$$\tilde{\delta}'(\beta) > 0.$$

Plugging $k = 0$, i.e., $\hat{\phi}(\phi, 1, k) = 1$ into (10), it immediately follows that

$$U = y.$$

If (11) fails, i.e., $\delta \in [\hat{\delta}, \tilde{\delta}(\beta))$, then it is optimal to set $\phi = 0$ to allow for the maximum δ^k (or minimum k). With $\phi = 0$, the maximum δ^k is achieved by $\delta^k \beta = \bar{\phi}$, or

$$\delta^k = \frac{\bar{\phi}}{\beta}.$$

Comparing the role payoff of the game

$$U = (1 - \delta) y + \delta \left(pU + (1 - p) \left(x + \delta^k (U - x)\right)\right)$$

with the player payoff

$$V = (1-\delta)\, y + \delta \left(pV + (1-p)\left(x + \delta^k \beta\,(V - x)\right)\right),$$

the fact that $\beta < 1$ implies that $U > V$. Also, the fact that $\delta^k < 1$ implies that $U < y$.

Plugging $\phi = 0$ and $\delta^k = \bar{\phi}/\beta$, i.e., $\hat{\phi}(\phi, 1, k) = \delta^k = \bar{\phi}/\beta$ into (10), we have

$$U = \frac{(1-\delta)\, y + \delta\,(1-p)\,(1 - \frac{\bar{\phi}}{\beta})x}{1 - \delta(p + (1-p)\,\frac{\bar{\phi}}{\beta})}.$$

■

According to Proposition 4.1, turnover of players creates Pareto improvement whenever players are patient enough such that some cooperation can be supported in the game without player turnover and that the incumbent player does not have 100% of bargaining power. When players are sufficiently patient, player turnover completely eliminates the inefficiency caused by imperfect monitoring. In fact this is true for any level of the incumbent player's bargaining power:

Corollary 4.1 (Folk Theorem) For all $\beta \in [0,1)$, there exist $\tilde{\delta}(\beta) < 1$ such that for all $\delta \geq \tilde{\delta}(\beta)$, there exists $\phi \in [0,1]$ such that the equilibrium action profile of the game is $a_t = EE$ for all t.

Proof. This follows immediately that $\tilde{\delta}(\beta) < 1$. ■

It is also shown that the Pareto improvement that can be achieved increases in players' level of patience and decreases in the incumbent player's bargaining power. In other words, the proposition points to an additional benefit of have more patient players playing a repeated game. The proposition also points out that in general the payoffs from a game is strictly higher than the payoffs to individual players, $\overline{U} > \overline{V}$.

5 Application 2: Relational Contract with Limited Liability

In this section, we analyze the example of relational contract with limited liability. This is a version of the model in 3.2.2. We look at the case when $M = \infty$. The purpose of this example is to illustrate how to apply the method developed in Subsection 3.3. In addition, we illustrate how the possibility of replacement affects the optimal relational contract.

Fong and Li (2008) analyze the same game without replacement. One key result there is that the optimal contract is never efficient. When replacement is allowed, we show that the output from the game is efficient.

While the results are different, the method of analysis here is similar to Fong and Li (2008). Therefore, we state several lemmas below without proof. Interested reader can refer to Fong and Li (2008) for more details.

As in Fong and Li (2008), we make the following two assumptions to simplify the analysis:

Asssumption 1:

$$py - \frac{[1-\delta(1-p)]c}{\delta(p-q)} \geq \underline{v} + \underline{w}.$$

This helps insure that the efficient outcome can be sustained as an equilibrium outcome.

Assumption 2:

$$\underline{w} \geq \underline{u}.$$

This says that the outside option of the agent is sufficiently low.

To characterize the PPE set, we use the method adapted in Section 3. Here, the state variable is the continuation payoff of the agent (in the game) and the principal. Denote $f(V)$ as the maximum PPE payoff of the principal if the agent's payoff is V. By Property 3 in 3.3.2, we see that f is well-defined and that it is cocave.

Denote V_e as the smallest PPE payoff of the agent that maximizes the sum of the payoff of the principal and agent. By assumption 1, the first best can be achieved, so V_e is also the smallest PPE payoff of the agent that achieves the first best. The first lemma states that, just as in Fong and Li (2008), the Pareto Frontier has a slope of -1 when $V \geq V_e$.

Lemma 5.1: *For* $V \in [V_e, V_e + f(V_e) - \underline{v}]$,

$$f(V) = f(V_e) + V_e - V.$$

Just as in Fong and Li (2008), define

$$k \equiv \frac{(1-\delta)c}{\delta(p-q)}$$

as the "bonus" necessary for the agent to put in effort. In other words, to induce the agent to put in effort, the difference in continuation payoffs should differ by at least k.

Also, define L as the unique linear operator such that

$$V = (1-\delta)(\underline{w} - c) + \delta[(1-p)L(V) + p(L(V) + k)].$$

For an agent with current normalized payoff V, $L(V)$ corresponds to the agent's continuation payoff next period if he is paid $\underline{w}$ this period, puts in effort, but the outcome is $Y = 0$. Alternatively, we can write

$$L(V) = \frac{V - (1-\delta)(\underline{w} - c)}{\delta} - pk.$$

The next lemma shows that we must have

$$L(V_e) = V_e.$$

Lemma 5.2:

$$\begin{aligned} V_e &= (\underline{w} - c) + \frac{\delta pk}{(1-\delta)} = L(V_e). \\ f(V_e) &= py - c - V_e. \end{aligned}$$

It is worth noting that, for $V \geq V_e$, the Pareto frontier of PPE in the game with replacement and without replacement is identical. However, as we will show below, the Pareto frontier of PPE with replacement is strictly larger when $V < V_e$. Before we start the analysis that gives this different outcome, we state another result that is shared in both replacement and no replacement.

Lemma 5.3: *For any PPE payoff on the Pareto frontier with* $V \in (\underline{u}, V_e)$, *the first period wage of the agent must be* $w_1 = \underline{w}$.

Now define V_r as the biggest V such that

$$f(V_r) \geq f(V) \text{ for all } V.$$

The key observation in the analysis of replacement is that, if $L(V) < V_r$, the best equilibrium for the principal is to run a lottery of termination, so that

$$L(V) = \phi \underline{u} + (1 - \phi) V_r,$$

where $\phi \in (0, 1)$ is the probability of termination. Through this operation, the principal can give the agent an expected payoff of $L(V)$ (and thus preserves the incentive for the agent.) In addition, the continuation payoff of the principal will be $f(V_r)$, the maximal payoff possible, regardless of the outcome of the lottery. This leads to the following lemma.

Lemma 5.4: *For* $V \in [V_r, V_e]$,

$$f(V) = (1 - \delta)(py - \underline{w}) + \delta[pf(L(V) + k) + (1 - p)f(\max\{V_r, L(V)\})] \qquad (12)$$

Proof. The proof follows in two steps. The first step shows that for $V \in [V_r, V_e]$, $(V, f(V))$ can be obtained by an equilibrium profile in which the first period play requires effort. This step is identical to Lemma 6 in Fong and Li (2008) and is omitted.

In the second step, suppose the agent's value is V. There are two cases to consider. In the first case, $L(V) < V_r$. In this case, V_L the expected payoff of the worker corresponding to $Y = 0$, $V_L \leq L(V)$. It then follows from the observation above that the principal's continuation payoff following a bad outcome can reach $f(V_r)$, which is the highest possible. In other words, if the agent isn't replaced after a high outcome, we have

$$f(V) = (1 - \delta)(py - \underline{w}) + \delta[pf(V_h) + (1 - p)f(V_r)],$$

where V_h are agent's payoffs corresponding to a high output (note that we don't need randomization here because of the concavity of f). Since $f(V) \leq f(V_r) < py - \underline{w}$, it follows that $V_h > V_r$. Since $f'(V_h) < 0$ and that $V_h \geq L(V) + k$, and it then follows that f is maximized by setting $V_h = L(V) + k > V_r$ if the agent is not replaced after a high outcome with probability 1.

If the agent is replaced after a hight outcome with some probability, let V_h be the expected payoff of the agent. Now suppose V_h is obtained through

$$V_h = \int V(x)\varphi(x)dx,$$

where $\varphi(x)$ is some (generalized) density function. If $V(x) \geq V_r$, there are two ways for the agent to receive $V(x)$. First, the agent can remain the relationship. In this case, the principal's continuation payoff next period is $V(x)$. Second, the agent can be paid a severance pay $V(x)$, and in the next period the principal will hire a new worker and get $f(V_r)$. Since

$$\begin{aligned} & -V(x) + \delta f(V_r) \\ < & -\delta(V(x) - V_r) + \delta f(V_r) \\ < & f(V(x)) \end{aligned}$$

where the last inequality follows from the fact that f is concave, so Lemma 1 implies $f' \geq -1$. It is then clear that for the principal's to maximize his continuation payoff, if the agent's continuation payoff is $V(x) > V_r$, the best way is to keep the agent.

Now let us define

$$\begin{aligned} g(x) &= f(x) \quad \text{for } x \geq V_r \\ &= f(V_r) \quad \text{for } x < V_r. \end{aligned}$$

By the concavity of f and the definition of V_r, it is clear that g is also concave.

Our discussion above suggests that for $V_h > V_r$

$$f(V_h) = g(V_h) \geq \int g(V(x))\varphi(x)dx$$

by Jensen's inequality, so to reach an average continuation payoff of V_h for the agent, the highest payoff of the principal is $f(V_h)$. Finally, since $f'(V_h) < 0$, we have that

$$f(V) = (1-\delta)(py - \underline{w}) + \delta[pf(L(V)+k) + (1-p)f(V_r)].$$

If $L(V) \geq V_r$, an anaglous argument as above gives that

$$f(V) = (1-\delta)(py - \underline{w}) + \delta[pf(L(V)+k) + (1-p)f(L(V))].$$

Putting the two together, we have (12). ■

The next lemma determines the exact value of V_r.

Lemma 5.5:

$$V_r = (1-\delta)(\underline{w} - c) + \delta \max\{(\underline{u} + pk), -(1-p)k\}.$$

Proof. It is clear $L(V_r) \geq \underline{u}$, because otherwise the agent will not put in effort.

Now if $L(V_r) > (1-\delta)(\underline{w} - c) + \delta \max\{(\underline{u} + pk), -(1-p)k\}$, then there exists a point $V_r - \varepsilon$ such that

$$\begin{aligned} L(V_r - \varepsilon) &> \underline{u}; \\ L(V_r - \varepsilon) + k &> V_r. \end{aligned}$$

Note that $V_r < V_e$, so we have

$$L(V_r - \varepsilon) < V_r.$$

It then follows that

$$\begin{aligned} f(V_r - \varepsilon) &= (1-\delta)(py - \underline{w}) + \delta[pf(L(V-\varepsilon)+k) + (1-p)f(V_r)] \\ &> (1-\delta)(py - \underline{w}) + \delta[pf(L(V)+k) + (1-p)f(V_r)] \\ &\geq f(V_r), \end{aligned}$$

where the first inequality follows from that fact that $L(V_r - \varepsilon) + k > V_r$ so $f'(L(V - \varepsilon) + k) < 0$. ■

The next lemma determines the value of f to the left of V_r.

Lemma 5.6: *For almost all* $V \in [\underline{u}, V_r]$,

$$f(V) = ((1-\delta)\underline{v} + \delta f(V_r)) + \frac{V-\underline{u}}{V_r - \underline{u}}(f(V_r) - \underline{v}).$$

Proof. For $V = \underline{u}$, the agent cannot put in effort in period 1, so the highest payoff for the principal is

$$f(\underline{u}) = (1-\delta)\underline{v} + \delta f(V_r),$$

implemented by replacing the agent with a new one at the beginning of period 2.

The concavity of f gives the expression for the rest of $V \in [\underline{u}, V_r]$. ■

If the value of $f(V_r)$ is known, Lemma 5.1-5.6 characterizes the equilibrium completely. The value of $f(V_r)$ is determined jointly by Lemma 5.4 and 5.6. In particular, just as in Fong and Li (2008), define T_z as an operator as follows. Let g be a function on $[V_r, V_e]$, then

$$\begin{aligned} T_z g(V) = &(1-\delta)(py - \underline{w}) \\ &+ \delta p(1_{\{L(V)+k<V_e\}} g(L(V)+k) + 1_{\{L(V)+k \geq V_e\}}(py - c - (L(V)+k)) \\ &\qquad + \delta(1-p)(1_{\{L(V)<V_r\}} Z + 1_{\{L(V)\geq V_r\}} g(L(V)) \end{aligned}$$

It is clear that T_Z is a contraction mapping, and let g_Z be the unique fixed point of T_Z. Then Lemma 5.4 and 5.6 imply that $f(V_r)$ is the unique value that satisfies

$$f(V_r) = g_{f(V_r)}(V_r).$$

As in Fong and Li (2008), we can show that the equation above induces a contraction mapping, and $f(V_r)$ can be found by applying standard techniques.

To summarize our results, we have the following:

Theorem 5.1: *The Pareto frontier of the PPE payoff satisfies*

$$f(V) = \begin{cases} ((1-\delta)\underline{v} + \delta f(V_r)) + \frac{V-\underline{u}}{V_r-\underline{u}}(f(V_r) - \underline{v}) & \text{if } V \in [\underline{u}, V_r] \\ (1-\delta)(py - \underline{w}) + \delta[pf(L(V)+k) + (1-p)f(\max\{V_r, L(V)\})] & \text{if } V \in [V_r, V_e] \\ f(V_e) + V_e - V & \text{if } V \in [V_e, V_e + f(V_e) - \underline{v}], \end{cases}$$

where $V_e = (\underline{w} - c) + \frac{\delta pk}{(1-\delta)}$, $f(V_e) = py - c - ((\underline{w} - c) + \frac{\delta pk}{(1-\delta)})$, $V_r = (1-\delta)(\underline{w} - c) + \delta \max\{(\underline{u} + pk), -(1-p)k\}$, *and* $f(V_r) = g_{f(V_r)}(V_r)$.

Just as in Fong and Li (08), when

$$\underline{u} + k = L(V_r) + k \geq V_e = (\underline{w} - c) + \frac{\delta pk}{(1-\delta)},$$

the PPE payoff can be calculated explicitly, but we omit it here. Instead, we list some of the properties of the optimal relational contract.

Proposition 5.1: *The optimal relational contract has the following properties:*

(i): Effort is put in in each period.

(ii): At the beginning of each employment relationship, the agent's normalized expected payoff is V_r*, which is weakly smaller than the agent's payoff in a game without replacement.*

(iii): The principal's payoff is higher than in a game without replacement.

(iv): When a new relationship starts, the agent is paid the minimum wage $\underline{w}$*. The agent is fired if his continuation payoff falls below* V_r*. The agent receives permanent employment when his continuation payoff exceeds* V_e*. In this case, the optimal contract can be sustained by a stationary contract.*

(v): The expected payoff of the agent is nondecreasing over time.

(vi): The expected payoff of the principal is nonincreasing over time in an employment relationship.

Proof. (i) follows directly from Lemma 4. The first part of (ii) follows from the proof of Lemma 5, and the second part follows from Fong and Li (2008). (iii) follows by noting that the operator in Lemma 4 is is "larger" than the operator in Lemma 6 in Fong and Li (2008). (iv) can be proved as Proposition 2 in Fong and Li (2008). (v) follows because of (ii) and that $\underline{w} < V_r$. (vi) follows because the principal's payoff is maximized at V_r, f is concave function, and that V_t is a submartingale. ■

Proposition 5.1 suggests that the pay of the worker is backloaded (c.f. Lazear (1981), Harris and Holmstrom (1982), and Ray (2001)). In the early stage of the employment relationship, the worker is paid the minimum wage, and the incentive of the agent is provided through the promise of permanent employment with good performance. After the agent attains permanent employment, he is incentivized by giving bonus for good performance. In fact, this structure is very similar to the case without limited liability as in Fong and Li (2008). However, there are some key differences as well.

First, full efficiency is not possible without replacement yet it is obtained here. This is because after an agent is terminated, the principal can find a replacement to resume the production. Perhaps at some level this is not surprising, but it reemphasize a general point of the paper: inefficiency within a bilateral relationship can often be ameliorated by replacement. More interestingly, ending a relationship is a bad news for the principal in the game without replacement. However, it is a good news here: the expected payoff of the principal is highest at the onset of a relationship. Perhaps this helps explain why many firms find their "seasoned workers" too expensive. But any attempt to fire these workers will only backfire, as, Schoonover, the CEO of Circuit City found out in a hard way, because it destroys the whole incentive system and leads to low morale.

Second, in a typical relational contract environment, the efficiency of the relational contract is nonincreasing with the principal's outside option. The possibility of replacement can be thought of as raising the principal's outside option; see for example Macleod and Malcomson (1989), Baker, Gibbons, and Murphy (1994, 2002, 2006), and Levin (2003). However, the Proposition above shows that the relational contract becomes more efficient with replacement. Moreoever, the principal's payoff also increases with replacement.

To resolve this puzzle, note that there are actually two kinds of outside options here. When a relationship dissolves, the outside option of the principal is affected by the nature how the relationship ends. If the relationship ends because the principal fails to pay the bonus or the principal fires the agent when he shouldn't, then the principal receives his "off-equilibrium outside option" $\underline{v}$. On the other hand, if the relationship ends because of the bad outcomes so the equilibrium prescribes a firing, then the principal receives his "equilibrium outside option" $f(V_r)$. The possibility of replacement affects the equilibrium outside option and not the off-equilibrium outside option.

If we extend the model so that replacement is not immediate but happens with positive probability β, then it can be shown that $f(V_r)$ increases with β. This contrasts with results in Ghosh and Ray (1996) and Kranton (1996). The difference here is that all of the outcomes are public observable, yet in the other papers, the new agents won't be able to observe what has happened in the past.

6 Application 3: Repeated Game without Transfer

In this section, we analyze the politician/citizen game. There are two main purposes of this example. First, we show that the efficiency of the game can change non-monotonically in the player's outside option, and this is illustrated in Subsection 6.1. In Subsection 6.2, we show that the efficiency of the game may be enhanced if players are allowed to re-enter the game. This calls to attention of the importance of design of institutions on entry-exit rules.

6.1 Efficiency and Outside Option

In this subsection, we analyze the politician/citizen game as set up in 3.2.1. We assume that, once a politician leaves the game, she/he will never be allowed to return. We focus our analysis on the maximal payoff that can be obtained by the citizen.

To characterize the Pareto frontier, we make two simplifying assumptions. Again define

$$k = \frac{(1-\delta)c}{(p-q)\delta},$$

which is the smallest difference in continuation payoffs necessary for the politician to exert effort.

Assumption 6.1:

$$y \geq \frac{1-p\delta}{1-\delta}k + x$$

This assumption makes sure that there are equilibrium that are sustained by (x, x) and (y, y) alone.

Assumption 6.2:

$$y + \frac{pk}{1-\delta} \leq \min\{z - \frac{x(z-y-pk)}{\delta y}, z - \frac{x(z-y-pk)}{y} - k\}$$

This assumption makes sure that there are equilibrium in which only (y, y) and $(z, 0)$ are played in the equilibrium.

With these two assumptions, we define L as the unique linear function that satisfies

$$\begin{aligned} V &= (1-\delta)y + \delta(L(V) + pk); \\ L(V) &= \frac{V - (1-\delta)y}{\delta} - pk. \end{aligned}$$

Define $f(V)$ as the maximal payoff of the citizen if the politician's payoff is V. Now we are ready to characterize the Pareto frontier of the PPE set. There are several cases to consider.

6.1.1 Low Outside Option

By low outside option, we mean that

$$\underline{u} < x.$$

In this case, the politician will never leave the game. Therefore, the analysis is identical to the case without replacement.

Define

$$V_x = y - \frac{\delta}{1-\delta}(1-p)k.$$

The lemma below shows that V_x is the biggest payoff of the agent among PPE payoffs that are played by (x, x) and (y, y) only.

Lemma 6.1: *For* $V \in [x, V_x]$,

$$f(V) = V.$$

Proof. Note that V_x satisfies

$$L(V_x) + k = V_x.$$

Assumption 1 guarantees that $V_x - k \geq x$. It is therefore clear that any payoff along (x, x) and (V_x, V_x) can be sustained as an equilibrium payoff. In addition, this is the upper-bound of the feasible payoff, so $f(V) = V$. ■

Now define

$$V_y = y + \frac{pk}{1-\delta};$$
$$V_z = z + x - \frac{x(z-pk)}{y}.$$

The lemma below shows that f is a straight line between V_y and V_z.

Lemma 6.2: *For* $V \in [V_y, V_z]$,

$$f(V_y) = \frac{(1-\delta)y(z-V_y)}{(z-V_y)(1-\delta)+\delta pk}$$
$$f(V) = \frac{V-V_y}{V_z-V_y}x + \frac{V_z-V}{V_z-V_y}f(V_y).$$

Proof. Consider the following dynamics on player continuation payoffs. The players start at $(V_y, f(V_y))$. The citizen and politician play (y, y). They stay at $(V_y, f(V_y))$ in the next period if the current period outcome is low and move to $(V_y+k, f(V_y+k))$ if the outcome is high. And $(V_y+k, f(V_y+k))$ is sustained by a randomization between $(V_y, f(V_y))$ and (V_z, x). In (V_z, x) shows up, then its first period play is $(z, 0)$, followed by a continuation payoff of

$$(\frac{V_z-(1-\delta)z}{\delta}, \frac{x}{\delta}),$$

which is again randomized between $(V_y, f(V_y))$ and (V_z, x). If the citizen deviates by playing x, then the game goes to (x, x) forever.

Assumption 2 guarantees that

$$V_y + k \leq V_z$$
$$V_y \leq \frac{V_z-(1-\delta)z}{\delta},$$

so the randomization is all interior, and thus above strategy can be supported as plays on the equilibrium path. Moreover, since all actions are either (y, y) or $(z, 0)$, this implies that $(V_y, f(V_y))$ lies on the line segment between (y, y) and $(z, 0)$. This gives the following equations:

$$f(V_y) = (1-\delta)y + \delta[(1-p)f(V_y) + pf(V_y+k)];$$
$$\frac{f(V_y+k)}{f(V_y)} = \frac{z-(V_y+k)}{z-V_y}.$$

Solving the equation, we have

$$f(V_y) = \frac{(1-\delta)y(z-V_y)}{(z-V_y)(1-\delta)+\delta pk}.$$

Finally, we can check that

$$\frac{f(V_y)}{x} = \frac{z-V_y}{z-V_z}.$$

And since the PPE payoff constructed above reaches the upper bound of feasible payoff, this gives the expression of f in $[V_y, V_z]$. ■

Finally, the following lemma determines the value of f for $V \in [V_x, V_y]$.

Lemma 6.3: *For* $V \in [V_x, V_y]$,

$$f(V) = (1-\delta)y + \delta[pf(L(V)) + (1-p)(f(L(V)+k)]$$

Proof. The proof method is very similar to Fong and Li (2008) and we omit it here. ■

Define operator T on bounded functions on $[V_x, V_y]$ as

$$\begin{aligned} Tg &= (1-\delta)y + \delta[p(1_{\{L(V)\geq V_x\}}g(L(V)) + 1_{\{L(V)<V_x\}}f(L(V)) \\ &\quad (1-p)(1_{\{L(V)+k<V_y\}}g(L(V)+k) + 1_{\{L(V)+k\geq V_y\}}f(L(V)+k)]. \end{aligned}$$

It is clear that T is a contraction mapping, so it has a unique fixed point. Then Lemma X.3 says that f restricted to $[V_x, V_y]$ is the fixed point of T, and thus it can be found by standard procedures.

Note that Lemma 6.1 and 6.2 have determined the value of f for $V \geq V_y$ and $V \leq V_x$, and with Lemma 6.3, we have the following:

Theorem 6.1: *When* $\underline{u} \leq x$, *the Pareto frontier of the PPE payoff is given by the following:*

$$f(V) = \begin{cases} V & \text{if } V \in [x, V_x) \\ g_T(V) & \text{if } V \in [V_x, V_y) \\ \frac{V-V_y}{V_z-V_y}x + \frac{V_z-V}{V_z-V_y}f(V_y) & \text{if } V \in [V_y, V_z], \end{cases}$$

where $V_x = y - \frac{\delta}{1-\delta}(1-p)k$, $V_y = y + \frac{pk}{1-\delta}$, $f(V_y) = \frac{(1-\delta)y(z-V_y)}{(z-V_y)(1-\delta)+\delta pk}$, $V_z = V_z = z + x - \frac{x(z-pk)}{y}$, *and* g_T *is the unique fixed point of* T *on bounded functions on* $[V_x, V_y]$ *such that*

$$\begin{aligned} Tg &= (1-\delta)y + \delta[p(1_{\{L(V)\geq V_x\}}g(L(V)) + 1_{\{L(V)<V_x\}}f(L(V)) \\ &\quad (1-p)(1_{\{L(V)+k<V_y\}}g(L(V)+k) + 1_{\{L(V)+k\geq V_y\}}f(L(V)+k)]. \end{aligned}$$

An immediate application of Theorem 6.1 is that, if the agent's outside option is so low that he cannot be induced to leave voluntarily, efficiency cannot be obtained.

Proposition 6.1: *When* $\underline{u} \leq x$, *for all* $V \in [x, V_z]$,

$$f(V) < y.$$

Proof. It is clear that $f(V) \leq y$. For $f(V) = y$, this requires that (y, y) is always played. It is clear that $f(V) < y$ for $V < V_x$ and $V > V_y$. Moreover, when V are in these regions, there are positive probability that (x, x) or $(z, 0)$ will be played. Now take any $V \in (V_x, V_y)$, in the dynamics of the continuation payoffs given by Lemma 3, there are positive probability that the continuation payoffs will hit the above two areas. So $f(V) < y$ for this region as well. ■

6.1.2 Higher Outside Options

By higher outside option, we mean that

$$\underline{u} \geq x.$$

In this case, it is possible to induce the politician to leave the game by threatening to play x forever. Now we characterize the Pareto frontier of the PPE and examine under what conditions efficiency can be obtained.

The analysis of characterizing the PPE frontier with replacement is very similar to those in Section 5, so we state the results and omit the proofs here.

Denote V_r as the politician's maximal payoff among those that maximizes the citizen's payoffs. Lemma 6.4 states that V_r is the larger value of either $L(V_r) = \underline{u}$ or

$L(V_r)+k = V_r$. This lemma is the direct analog of Lemma 5.5 in the case of relational contract with limited liability.

Lemma 6.4:

$$V_r = \max\{(1-\delta)y + \delta(\underline{u} + pk), y - \frac{\delta(1-p)k}{1-\delta}\}.$$

Now again let $V_y = y + \frac{pk}{1-\delta}$ and $V_z = z + x - \frac{x(z-pk)}{y}$. The lemma below shows that, just as in the case with low outside option, f is a straight line between V_y and V_z. This lemma is an analog of Lemma 5.6.

Lemma 6.5: *For* $V \in [V_y, V_z]$,

$$\begin{aligned} f(V_y) &= \frac{(1-\delta)y(z-V_y)}{(z-V_y)(1-\delta)+\delta pk} \\ f(V) &= \frac{V-V_y}{V_z-V_y}x + \frac{V_z-V}{V_z-V_y}f(V_y). \end{aligned}$$

Finally, the next lemma states that for $V \in [V_r, V_e]$, f satisfies a functional equation similar to Lemma 5.4 in the analysis of limited liability.

Lemma 6.6:

$$f(V) = (1-\delta)y + \delta[pf(L(V)+k) + (1-p)f(\max\{V_r, L(V)\})]$$

Again, the right hand side of Lemma X.3 defines a contraction mapping, so f can be found by using standard methods. Therefore, lemma X.1-X.3 characterizes the PPE frontier completely.

This characterization allows us to state conditions under which efficiency can be obtained. This is the content of the next proposition.

Proposition 6.2: Efficiency of the game can be obtained if and only if

$$y - \frac{(1-\delta p)k}{1-\delta} \geq \underline{u}.$$

Proof. Denote V_{In} as the largest V in which $f(V) = y$. Then we have

$$V_{In} = (1-\delta)y + \delta(pV_{In} + (1-p)(V_{In} - k))$$

because otherwise $L(V_{In}) + k > V_{In}$ and we must have $f(V) < y$. This gives that

$$V_{In} = y - \frac{\delta(1-p)k}{1-\delta}.$$

Now to sustain V_{In} as an equilibrium, we must have

$$V_{In} - k \geq \underline{u}.$$

This is exactly the condition in the proposition. ■

It is also easy to show that as $\underline{u}$ increases further, the maximal payoff of the citizen decreases. And the equilibrium collapses when

$$\underline{u} \geq L(V^*),$$

and

$$\frac{y}{z-y} = \frac{x}{z - V^* - k}.$$

Finally, it is easy to show that if $\underline{u}$ is larger than $y - \frac{(1-\delta p)k}{1-\delta}$, then the effciency of the game, characterized by the payoff of the principal, decreases with $\underline{u}$. The proof of it is similar to Proposition 3 in Fong and Li (2008), and we omit it here.

Proposition 6.3: Let $F(\underline{u}) = \max_V f(V)$ when the politician's outside option is $\underline{u}$. For $\underline{u} > x$,

$$\frac{dF(\underline{u}}{d\underline{u}} \leq 0.$$

6.2 Reentry

In the previous subsection, we assume that once the politician leaves the game, he cannot return. We found that if $\underline{u} < x$, then the play is inefficient in the sense that $f(V) < y$ for all V. In this subsection, we show that if the players are allowed to reenter, then the efficiency may be obtained even if $\underline{u} < x$.

Assume that there are $1 < M < \infty$ politicians in total. We also assume that if the current politician leaves the game this period, any of the M politician (including the exiting one) has equal chance of entering the game. Other transition functions are possible, but the qualitative results does not change.

Proposition 6.3: Efficiency of the game can be obtained if and only if

$$y - \frac{(1-\delta p)k}{1-\delta} \geq \underline{u} + \frac{y - \frac{\delta(1-p)k}{1-\delta} - \underline{u}}{\delta + M(1-\delta)} \geq x.$$

Proof. Denote V_{In} as the largest V in which $f(V) = y$. Then again we have

$$\begin{aligned} V_{In} &= (1-\delta)y + \delta(pV_{In} + (1-p)(V_{In} - k)); \\ V_{Out} &= \frac{1}{M}V_{In} + \frac{M-1}{M}((1-\delta)\underline{u} + \delta V_{Out}). \end{aligned}$$

This gives that

$$\begin{aligned} V_{In} &= y - \frac{\delta(1-p)k}{1-\delta}; \\ V_{Out} &= \underline{u} + \frac{V_{In} - \underline{u}}{\delta + M(1-\delta)}. \end{aligned}$$

For the above to be an equilibrium, we need

$$\begin{aligned} V_{Out} &\geq x; \\ V_{In} - k &\geq V_{Out}. \end{aligned}$$

This is exactly the condition in the statement of the proposition. ■

Note that the inequality in Proposition 6.3 may be satisfied even if $\underline{u} < x$. This is because the possibility of reentry increases the de facto outside option of the politician. Even if the immediately payoff outside game may be lower, the possibility of being able to reenter makes the politician prefer leaving the game to staying in the game and forever receiving a low payoff x.

On the other hand, Proposition 6.3 also states if reentry is too easy (M is small), then efficiency again becomes difficult to sustain. This is because when the number of potential politicians becomes small, this raises the outside option of the politician, and this makes it difficult to induce effort from the politician. Therefore, efficiency can be obtained if and only if the number of potential candidates is in the intermediate range.

7 Conclusion

In this paper, we proposed a simple theory of player turnover in repeated games according to which player turnover takes place because it allows the suspected deviator to get punished without punishing the long-term relationship. Although our theory of turnover is rather general for that it encompasses voluntary and involuntary turnover, turnover allowing and disallowing payments to departing players, there are various extensions which can lead to other applications. Here we discuss a few of them.

We currently restrict to repeated games with imperfect public monitoring. In Fong, Li, and Matouchek (2008), we analyze turnover in a repeated game of private monitoring and apply it to develop a theory of CEO turnover.

In our current setup, the number of players in a repeated game is fixed. Another useful extension is to allow for flexible number of players in a relationship. One application of such extension is one of experience goods sellers in a general equilibrium setup. When disappointed buyers of experience goods can switch from one firm to the other, we conjecture that firms' incentives can be properly discipline without aggregate punishment.

Although in our general setting we allow the number of potential players in the global game to be arbitrary, in all the examples we investigate, we assume there are infinitely many replacement players so a replacement player is always available. One natural extension is to model probabilistic availability of replacement players. We also assume that there is no exogenous cost associated turnover of players. It will be interesting to study a more realistic setting in which turnover is costly and analyze how the cost of turnover impact the equilibrium payoff set and strategies that support these payoffs.

Although we study both voluntary and involuntary turnover, the right to replace a player is exogenously determined in our setup. One interesting extension is to endogenize the protocol of termination of relationship as part of the optimal relational contract. In general, an optimal relational contract should specify whether the principal has the right to replace the worker or whether the worker can only be replaced when he voluntarily resigns.

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