Unhedgeable Gamma Risks in Asset Pricing.

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Unhedgeable Gamma Risks in Asset Pricing.

Lévy processes extend the diffusion framework by accommodating extreme local activity, while incorporating pure jump fundamentally unhedgeable components. The differential intensity of precautionary savings for a Lévy process is related to jump risk and motivates the definition of a sensitivity that captures inability to hedge its non-diffusive risks. Unlike diffusions, gamma processes are scaled under Q by their sensitivity. Furthermore, the sensitivity ratio determines the risk neutral drift reduction. I show that total sensitivity of composite risks equals an $R^2$-weighted sum of the individual sensitivities. The inability to hedge a time changed Brownian motion is discussed. Catastrophic risks are inflated by their sensitivity while booming risks are deflated.
Diffusion based models are characterized by an almost certain path continuity. If only diffusive risks are present, then such path continuity, coupled with the assumed ability of agents to continuously trade, ensures that agents choose their optimal trading strategy facing only small local risks. Diffusion risks can be perfectly hedged because their continuity implies that after a very short interval prices haven’t moved much, and the associated rebalancing is infinitesimal. But if on infinitesimally short intervals random jumps happen, the agent will have to substantially rebalance. In reality, less than perfect liquidity forces agents to incorporate unhedgeable risks in their decision making. One of the reasons that liquidity, a seemingly simple concept of our ability to trade large quantities quickly, and at a low cost, has been elusive from theoretical modelling is that path continuity renders diffusion driven assets infinitely liquid.

Traditionally, following Merton (1976), in order to incorporate discontinuity in the price process a diffusion is enhanced with the introduction of Poisson jumps, that is rare and large breaks in the process. Even though such jump-diffusion models can correctly incorporate extreme events at a macroeconomic level, their low jump activity cannot suitably capture dis-
continuity at a microstructure level. Alas, exactly such “local” discontinuity, along with transaction costs, limits agents’ ability to completely hedge, and is fundamentally related to the liquidity of the underlying market.

One of the most documented failures of the diffusion model emerge in the puzzling deviations of out of the money option prices from the Black-Scholes predictions. Such options are the contracts that provide insurance against those states where the trading process is expected to exhibit the greatest deviations from the perfectly liquid paradigm.

It is conceivable that a richer structure of uncertainty, that transcends the Brownian frame of reference, explains agents’ ex ante consideration for their ability to hedge, and, has profound asset pricing implications which are not captured by the diffusion based models. Such considerations have recently generated research activity that aims to incorporate infinite activity jump risk in asset pricing. Identifying and disentangling the two types of risk is of interest to traders and risk managers who can then apply diffusion techniques in dealing with the Brownian component, while evaluating tail risk (e.g. with VaR methods) for the jump component.
A Lévy process extends the Brownian motion by incorporating jumps that arrive at an infinite rate. The jump arrivals of a Lévy process generate infinite activity, which is traditionally handled by the diffusion component in a jump-diffusion setting. Central examples of infinite jump activity models are the normal inverse Gaussian by Barndorff-Nielsen (1998), and the hyperbolic by Eberlein, Keller and Prause (1998). Carr and Wu (2003a) motivate the use of a log-stable model for option pricing from the observation that the implied volatility smirk does not flatten out, and even slightly steepens as maturity grows. In a more general framework, Carr and Wu (2003b) study option pricing for the general class of time-changed Lévy processes.

Time changes are fundamental for modern asset pricing since, as is argued in Carr and Wu (2003b), time changed Lévy processes can capture stochastic volatility and leverage effects simultaneously. Geman, Madan and Yor (2001) argue that the market practices of including jumps and stochastic volatility extend a Brownian motion by changing its time. By recognizing that most return processes of interest are Brownian motions evolving at a random clock they argue that even though the pure jump component in the returns’ dynamics is clearly needed, the same is not necessarily true for the
diusion component. Geman, Madan and Yor (2002) address what they call the recovery issue, i.e. how much we may learn about the time change itself by only observing the price process. Further, by time-changing a homogeneous Lévy process stochastic volatility is induced. Carr, Geman, Madan and Yor (2003) use this method and formulate twenty-five models in order to estimate parameters that parsimoniously capture the volatility surface and can be used as the state of the system for option pricing.

On the issue of estimating the parameters of Lévy processes Ait-Sahalia (2004) finds that in a mixture of Brownian motion with a pure jump Cauchy process maximum likelihood estimators can still asymptotically distinguish the two sources of risk. Carr and Wu (2003c) propose a method to distinguish among continuous and discontinuous components in the price process by studying the speed at which ATM option prices tend to zero as they expire. Huang and Wu (2004) empirically analyze option pricing models based on time changed Lévy specifications.

Broadly speaking, my objective in this paper consists in developing a rational concept that can measure "how unhedgeable" a specific Lévy process is for asset pricing applications. Once adopted, such a measure may help
market participants by summarizing the type of infinite activity exhibited by a Lévy process in a way that can directly be employed in asset pricing. Of course one may argue that all this information is already in the Lévy measure of the process. True, but the Lévy measure may be too informative in the sense that it is not easy for the non-specialist to digest, and understand the difficulty in dealing with a specific Lévy process. In a sense, I want to construct a method that distills the entire Lévy measure to a single number that reflects the degree of unhedgeability.

The first task of the paper is to study the implications of risk aversion in the context of Lévy processes. The known theory establishes that yields are determined by the time value of money, and, the precautionary saving motive of agents to hoard assets away for future consumption in an uncertain environment. It has been known since Leland (1968) that precautionary savings in response to risk are associated with the convexity of the marginal utility. Rothschild and Stiglitz (1971) and Kimball (1990) show that when an individual’s utility depends on an exogenously uncertain outcome, and his control variable $C$, and the marginal utility is convex, an increased uncertainty will result in a decreased consumption $C$. The intuition is that
an increased variance will further magnify the positive Jensen effect, due to the convexity of the marginal utility, and the agent will have to lower \( C \) in order to equalize the first and second period marginal utilities. Such a precautionary saving motive, that is the propensity of prudent agents to save more when faced with increased uncertainty, will lower interest rates. In the traditional diffusion setup this negative relation between the risk free rate and risk, \( \frac{\partial r_f}{\partial \sigma} < 0 \), equals \(-\sigma \vartheta^2\), where \( \vartheta \) is the risk aversion parameter, and is understood as a direct result of aversion to volatility.

In a more general Lévy framework, where risks may contain pure jump unhedgeable components, lower risk free rates will not only capture risk aversion but will also relate to the inability to completely hedge. It is then important to disentangle the two effects, namely pure aversion to risk -which is present even in the pure diffusion case- from inability to hedge -that is only present when the pure jump component exists.

In the first part of the paper, the intensity of the precautionary saving motive is related to the type of underlying Lévy risk by measuring how unhedgeable volatility generated by a Lévy process affects risk free rates. A properly defined volatility sensitivity \( \varphi \) of an economy is proposed as a
measure of the inability to hedge. By definition the $\varphi$ ratio measures the differential volatility sensitivities of diffusive versus non-diffusive processes. The total inability to hedge composite Lévy risks is shown to equal an $R^2$-weighted sum of the individual risk $\varphi$ factors.

In the second part the central example where dynamics are driven by a gamma process is worked out in detail. The gamma process is an important case since its powerful analytical properties allow empirical investigations to directly incorporate Lévy risk. Furthermore, the gamma is not only interesting by itself but even more as a subordinator that changes time in a Brownian motion. The economic significance in this context is in allowing transaction time, or business activity, to evolve at a rate different from calendar time.

Heston (1993) studied the pure gamma return generating process in the context of option pricing. A diffusion with its time changed by a gamma is called a Variance gamma (VG) process. Madan, Seneta (1990), Madan, Milne (1991), developed the symmetric VG process, while Madan, Carr and Chang (1998) employed an asymmetric version of the VG process to derive option prices. Since its introduction the VG process has gained momentum in a variety of applications and has even been implemented in investment
banks with daily estimated parameters as Geman, Madan and Yor (2002) report.

Unlike a Brownian motion, a pure gamma always has a non-zero drift. This can cause a confusion when in manipulating the parameters of the gamma to calibrate higher moments one is also indirectly changing the drift. The novelty of the paper is that in order to effectively deal with this issue, and be able to measure the true volatility sensitivity of a process, the drift of the original gamma process is removed. Thus in this paper, besides volatility $\sigma$, the other natural parameter of the gamma becomes skewness $\varsigma$ instead of the usual drift.

A central finding of the paper is that, unlike Brownian volatility, which remains invariant, risk neutral volatility of unhedgeable gamma risks is scaled by their volatility sensitivity, i.e. $\sigma^* = \varphi \sigma$. It is shown that the unhedgeable risk factor $\varphi$ depends on both volatility and skewness. When the gamma risk is booming (i.e. driven by positive jumps), $\varphi$ is less than one. When the gamma risk is catastrophic, $\varphi > 1$. The intuition gained in this paper by studying the gamma process provides some new insights into how time changed risks are priced as well. A VG process which is defined as a gamma
time-changed Brownian motion is equal to the difference of two gamma processes (e.g. Madan, Carr, and Chang 1998). Then, because of volatility scaling, for a VG process the booming component will be deflated while the catastrophic component inflated under Q. Since the booming gamma generates positive skewness while the catastrophic gamma generates negative skewness, under Q the variance gamma process will have more negative skewness. This is related to the non-parametric approximate result in Bakshi, Kapadia and Madan (2003).

The paper is organized as follows. In the first section, the concept of precautionary savings naturally motivates the definition of the volatility sensitivity ratio for Lévy economies. In the second section the sensitivity of a gamma process is measured. In the third section the theory is extended to incorporate multiple risks. The inability to hedge a time changed Brownian motion is treated with this technique.
I Discounting Lévy risks

Since my desire is to focus on the volatility sensitivity of asset pricing, and in order to avoid difficult issues relating to how risk aggregates in a heterogeneous economy, a partial equilibrium with consumption dynamics exogenous to the price formation process will be modelled. More specifically, utility maximizing behavior is modeled in a simple pure exchange economy, in the framework provided by Lucas (1978). Consider a representative agent maximizing the utility produced by a consumption stream $C_t$,

$$\max E \int_0^\infty e^{-\delta t} u(C_t) dt$$

The price dynamics $S_t$ of a non-dividend paying stock have to satisfy the usual first order conditions,

$$S_o = E e^{-\delta t} \frac{u'(C_t)}{u'(C_o)} S_t$$

Assume power utility,

$$u(C) = \frac{C^{1-\vartheta}}{1-\vartheta}, \quad \vartheta > 0$$

where risk aversion is given by $\vartheta$. In this case only consumption growth matters, and thus, with no loss in generality we can assume that initially $C_o = 1$. 

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Uncertainty in the economy is modelled as a real-valued stochastic process $x_t$ with $x_0 = 0$. I assume that $x_t$ is a Lévy process with respect to the underlying filtration, i.e. $x_{t+\tau} - x_t$ is independent of the current state, $x_t$, and is distributed as $x_\tau$. In asset pricing applications, tractability is achieved by restricting the universe of Lévy processes to those with an analytic (at zero) characteristic function, so that, moments and cumulants are well defined.\footnote{The analyticity of the characteristic function at zero implies the existence of the moment and cumulant generating function $K(s)$ (see Lukacs (1970) ch. 7).}

The critical property of a Lévy process, that directly emerges from the definition, is that its moment generating function, when well defined, attains a special form in which time gets factored out,

$$Ee^{sx_t} = e^{tK(s)}$$

where $K(s)$ is the cumulant generating function.

In the analysis it is helpful to separate the deterministic growth from the stochastic component. That is, with no loss in generality, I assume that $E x_t = 0$ and growth $c_t = \ln C_t$ is driven by

$$c_t = \ln C_t = \mu t + x_t$$

where $\mu$ is equal to the growth rate. The equilibrium Euler condition (1)
translates to

$$S_0 = e^{-\delta t - \vartheta \mu t} E e^{-\vartheta x_t} S_t = E e^{-\xi_t} S_t$$  \hspace{1cm} (4)

where

$$\xi_t = \delta t + \vartheta \mu t + \vartheta x_t$$  \hspace{1cm} (5)

is the discount rate process.

A Risk and the Risk Free Rate

In an uncertain environment, prudent agents ($u'' > 0$) will always save more to insure themselves against bad future states. The economic intuition behind precautionary savings is that the convexity of the marginal utility, $u'' > 0$, generates a positive Jensen effect that agents can only match by lowering their consumption today. Kimball (1990) defines the concept of compensating precautionary premium $\Delta W$ as the dollar amount of extra wealth an agent should have so that he still consumes the same today even in the face of uncertainty. In Kimball’s two period economy where the agent solves

$$\max_c u(C) + E u(W - C + X)$$
and $X$ is a zero mean shock that captures uncertainty, $\Delta W$ is defined by:

$$Eu'(W - C + \Delta W + X) = u'(W - C)$$

The general theory of prudence does not provide a measure of the dependence of precautionary savings on the type of risk. The following lemma establishes the exact functional dependence of precautionary savings on the type of Lévy risk through its cumulant function,

**Lemma 1** *Lévy type uncertainty lowers risk free yields,*

$$\frac{\partial r_f}{\partial \sigma} < 0$$

and furthermore the precautionary saving effect on risk free rates equals $K(-\vartheta)$.

**Proof:** Let us assume that CRRA agents face a deterministically growing economy at a rate $\mu$ (i.e. $C_t = e^{\mu t}$). In this case the risk free yield only has to compensate them for their time preference $\delta$, and the opportunity cost of not investing in equity that grows at a rate $\mu$. The risk free yield will thus equal their time preference plus the risk aversion scaled growth,

$$r^f = \delta + \vartheta \mu$$
In the uncertain case discount rates $\xi_t$ are stochastic and the risk free rate will have to satisfy (4):

$$E[e^{-\xi_t e^{rt}}] = 1 \Rightarrow r^f = \delta + \vartheta \mu - K(-\vartheta) \quad (8)$$

where $K(.)$ is the cumulant generating function of the Lévy process, defined in (2). The term $K(-\vartheta)$ is thus, clearly, related to precautionary savings. The function $K$ goes through the origin while its first derivative $K'(0)$ is equal to the drift of $x_t$ and thus equals zero (see figure 1). Finally $K(.)$ is strictly convex which implies that the quantity $K(s)$ is positive for all $s \neq 0$ in its domain,

$$K(-\vartheta) > 0 \quad (9)$$

It is then clear that uncertainty lowers interest rates. ■

In the case of Lévy risk a new technical intuition behind lemma (1) emerges from this proof. Variance determines the convexity at zero of the $K(s)$ function. An increased variance will thus increase the convexity, and result in a higher value for $K(-\vartheta)$. Furthermore, observe that in such Lévy economies we can refer to $r^f$ in (8) as the risk free rate without any ambiguity, since, risk free yields do not depend on maturity (i.e. the term structure is

\[\text{The known convexity of cumulant functions is shown using Hölder's inequality.}\]
B The Volatility Sensitivity Ratio

Equation (8) is an important finding because it captures the exact savings effect for all Lévy processes, and is a generalization of the analogous relation for single factor log-normal economies originally developed by Hansen and Singleton (1983). In the case of log-normal consumption,

\[ x_t = \sigma B_t \]  

(10)

where \( B_t \) is a standard Brownian motion,\(^3\) relation (8) specializes to the well known

\[ r^f = \delta + \vartheta \mu - \frac{1}{2} \vartheta^2 \sigma^2 \]  

(11)

In the traditional diffusion setup the sensitivity \( \frac{\partial r^f}{\partial \sigma} \) equals \(-\sigma \vartheta^2\), and is recognized as a direct result of exposure to volatility.

In a more general Lévy framework, where risks may contain pure jump unhedgeable components, such risk free rate lowering will not only capture aversion to risk but will also compensate for the inability to completely hedge.

\(^3\)Remember that in (3) the drift \( \mu \) has been removed from \( x_t \).
It is then important to disentangle the two effects, namely pure aversion to risk -which is present even in the pure diffusion case- from inability to hedge -that is only present when the pure jump component exists. The motivation behind the definition of the volatility sensitivity ratio is that while the negative sign of the derivative (6) clearly establishes a precautionary saving motive, its magnitude will depend on the specific risk. More specifically, the relative magnitude of (6), vis-a-vis a purely diffusive economy, will depend on how confident the agent feels about hedging risks. I consider the ratio of the sensitivity (6) to the sensitivity of a purely diffusive economy with the same volatility,  
\[
\frac{\partial r_t}{\partial \sigma} = -\sigma \psi^2.
\]

**Definition 1** Define the volatility sensitivity ratio, \( \varphi \), of a Lévy process as the ratio of its sensitivity \( \frac{\partial r_t}{\partial \sigma} \) to the purely diffusive sensitivity \(-\sigma \psi^2\).

Indirectly, the volatility sensitivity ratio measures the strength of the propensity to save, due to an increase in volatility, relatively to an economy where only diffusive risk is present. The intuition of the definition is that the differential intensity of the precautionary saving motive when agents are exposed to risk generated by a Lévy process should capture their inability to hedge its pure jump component.
An immediate consequence of the definition establishes the units in which the $\varphi$ ratio is measured since the volatility sensitivity ratio in an economy driven by a Brownian motion is normalized to one. I will thus say that an economy with $\varphi$ ratio has $\varphi$ times the sensitivity of a purely diffusive economy.

C The Risk Neutral Measure

When agents are risk averse the empirical return distributions differ from the risk neutral ones. It is shown in the appendix that for CRRA agents, the risk neutral process is a member of the exponential family of the original process $x_t$. The negative risk aversion parameter becomes the \textit{tilting} parameter that connects the two measures:

$$\left( \frac{dQ}{dP} \right)_t = e^{-\vartheta x_t - tK(-\vartheta)}$$

(12)

Given the Radon-Nikodym derivative (12), the risk neutral density will be characterized by a cumulant function\footnote{In the paper, a star will always denote risk neutral quantities.} $K^*(s)$ which results from a first difference of the actual $K(s)$:

$$K^*(s) = \frac{1}{t} \ln E^* (e^{sx_t}) = K(s - \vartheta) - K(-\vartheta)$$

(13)
The cumulants of a Lévy process are the time-scaled derivatives at zero of its cumulant generating function. An important consequence then, is that the empirical cumulants of $x_t$ are recovered with differentiation of $K(s)$ at $s = 0$, while the risk-neutral cumulants are recovered by differentiation of the same function at the point $s = -\vartheta$. For example, while $E x_t = tK'(0) = 0$, the risk neutral mean is given by $E^* x_t = tK'(-\vartheta) < 0$ (see also figure 1).

C.1 Hedging and the risk free rate.

It is well known that in the case of time separable power utility, risk aversion determines not only curvature across different states but also the elasticity of intertemporal substitution. So the level of risk aversion impacts risk free rates via two counter-acting forces. Increasing the risk aversion makes agents less elastic to substitute consumption today for tomorrow and thus raises required yields. On the other hand, a higher risk aversion leads to a greater desire to hedge and thus tends to lower rates. As is shown in the next lemma, in a Lévy setting the total dependence of interest rates in risk aversion is captured by the usual drift lowering under $Q$.

**Theorem 1** The risk aversion sensitivity of the risk free rate equals the risk
corrected drift of economic growth,

\[ \frac{\partial r^f}{\partial \vartheta} = \frac{1}{t} E^* c_t = \mu^* \]

**Proof:** Risk neutral expected economic growth equals:

\[ E^* c_t = \mu t + E^* x_t = \mu t + K'(-\vartheta) t \]

From (8) we see that \( \frac{\partial r^f}{\partial \vartheta} = \mu + K'(-\vartheta). \)

I have already shown that the risk neutral drift of \( x_t \) is negative, \( K'(-\vartheta) < 0 \). This negative component is related to hedging across states and is responsible for the lowering of the risk neutral drift. For a Brownian motion with volatility \( \sigma \) the cumulant function equals \( K(s) = \frac{1}{2} s^2 \sigma^2 \) and risk neutral drift, \( K'(-\vartheta) \), equals \( -\sigma^2 \vartheta \).

**C.2 Skewness and the risk free rate.**

In the same spirit that higher volatility leads to lower rates through the precautionary savings mechanism, one suspects that positive skewness, generated by positive jumps, will lead to higher rates by improving the future
economic outlook, and, leading agents to bolder positions. In a sense, agents are more than happy to participate in an unhedgeable booming economy. Inversely, the negative skewness generated by a negatively jumping process will lower rates. Agents are scared by the possibility of unhedgeable catastrophic jumps and fly to the safety of money market accounts.

The mechanism of how this simple economic intuition—of the linkage between negative skewness and lower interest rates—formally works is best understood by studying figure 1. In this figure, I plot the cumulant function for a symmetric Brownian motion, and pure jump gamma processes for various values of skewness, while keeping volatility constant. Since, by construction, these are zero drift processes, the slope of their cumulant function at the origin is equal to zero.

Keeping volatility constant while incorporating negative skewness (via negative jumps) imparts a greater convexity for negative values on $K(s)$. This occurs because convexity at zero is fixed ($K''(0) = \sigma^2$) while negative skewness implies a negative third derivative and thus a decreasing second derivative of the cumulant function.

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5Even though Brownian and Gamma processes are used to plot $K(s)$ in figure 1, the discussion here does not depend on the specific Lévy processes but only on universal properties of their cumulant functions.
derivative which determines convexity. The higher convexity for negative values implies that an increased negative skewness decreases the risk free rate by increasing $K(-\vartheta)$.

Finally, observe that risk neutral drift is lowered by $K'(-\vartheta) < 0$. An increasingly negative skewness will also make $K'(-\vartheta)$ more negative and further lower the risk neutral drift.

II The gamma process

The core of the model is a pure jump gamma process, $x_t = \gamma_{\nu t}$ for $\nu > 0$. A Lévy process is defined by the distribution of its independent increments over non-overlapping intervals. For the time interval $(t, t+\tau)$ the increment $x = \gamma_{\nu(t+\tau)} - \gamma_{\nu t}$ of the process is independent of the current state, $\gamma_{\nu t}$, and is distributed as a gamma variate with $\nu\tau$ degrees of freedom,

$$f(x) = \frac{e^{-x}x^{\nu\tau}}{\Gamma(\nu\tau)}$$

(14)

As with all Lévy processes we assume that $x_0 = 0$.

The next step in the construction of the $x_t$ process is to enrich the single-
parametric family of $\gamma_{\nu t}$ processes by a scale parameter $\lambda$, as in $x_t = \lambda \gamma_{\nu t}$.

When the process is determined by its scale $\lambda$ and degrees of freedom $\nu$ parameters, its drift equals

$$\varphi = \lambda \nu$$

(15)

and its variance rate is given by

$$\sigma^2 = \lambda^2 \nu$$

(16)

The Lévy character of the process is due to the fact that the degrees of freedom of independent gamma variates are additive. More specifically the unique Lévy-Khintchine representation of the log characteristic function of $x_t = \lambda \gamma_{\nu t}$

$$\ln E e^{iux_t} = -\nu t \ln(1 - i\lambda u)$$

does not contain a quadratic term and thus the process has no diffusion component.

Similar processes\(^6\) are used, in the construction of the Variance gamma, in Madan and Seneta (1990), Madan and Milne (1991), and Madan, Carr and Chang (1998) to explain the passage of time and thus drive the variance.

\(^6\)In the Variance Gamma context the gamma process is generally constrained to a unit drift, $\varphi = \lambda \nu = 1$. 

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of returns, and in the context of option pricing by Heston(1993). The gamma process is pure jump, with an infinite arrival rate of jumps most of which are small. The small size and infinite arrival rate of jumps generates extreme local activity reminiscent of a diffusion but with discontinuous paths. For \( \lambda > 0 \), the Lévy measure of the process,

\[
\rho(dx) = \nu \frac{e^{-x/\lambda}}{x} dx \quad \text{for } x>0
\]

reveals the infinite arrival rate and the concentration of jumps around zero. Because of its pure jump character, unlike a diffusion that can be approximated by a binomial tree, the infinitesimal change in a gamma process can take infinitely many values and is thus fundamentally unhedgeable in the sense that trading a finite set of assets does not complete the markets.

As can be seen from (15), unlike a Brownian motion, a pure gamma always has a non-zero drift. This can cause a confusion when in manipulating the parameters of the gamma to calibrate higher moments one is also changing the drift. To bypass this issue, and conform with the general case (3), the drift of the original gamma process is removed,

\[
x_t = \lambda \gamma_{\nu t} - \lambda \nu t
\]

When \( \lambda > 0 \) the jumps of \( x_t \) are always small but positive and lead to positive
skewness. To enhance intuition, I will sometimes call such a risk **booming**.

In a sense, the constant deterministic decline, at a $\lambda \nu$ rate, of the process is constantly interrupted by tiny positive explosions arriving at an infinite rate that keep expectation at zero.

On the other hand, there are situations where a negatively skewed process needs to be employed. Since a booming gamma process always jumps up, a negative gamma process ($\lambda < 0$) will always jump down. A negatively skewed gamma risk will be called **catastrophic**. The cumulant function of the $x_t$ process is now equal to

$$K(s) = -\lambda \nu s - \nu \ln(1 - \lambda s)$$

and the risk free rate (8) will thus be given by

$$\rf = \delta + \vartheta \mu - \lambda \nu \vartheta + \nu \ln(1 + \lambda \vartheta)$$

$$= \delta + \vartheta \mu - \lambda \nu \vartheta - \nu \ln \varphi$$

where

$$\varphi = (1 + \lambda \vartheta)^{-1}$$

is a positive scaling factor that has central importance in capturing our inability to hedge.
Corollary 1  In equilibrium $\varphi > 0$.

Proof: From $\varphi = (1 + \lambda \vartheta)^{-1}$, it is obvious that when the gamma risk is booming, $\lambda > 0$, $\varphi$ is always well defined and positive. For economies driven by a catastrophic gamma, $\lambda < 0$, that have reached an equilibrium, the idea is that for a given risk aversion $\vartheta$, there is a negative singular value for $\lambda$ equal to minus the inverse risk aversion. If $\lambda$ is close to $-1/\vartheta$, risk free rates (20) will tend to minus infinity. Thus for an equilibrium with a positive and finite supply of the risk free assets to exist, $\varphi$ has to attain a real positive value. In a sense, for asset pricing to be conducted $\varphi$ has to be positive. ■

To show that $\varphi = (1 + \lambda \vartheta)^{-1}$ is indeed the sensitivity of the gamma process, $r^f$ has to be expressed as a function of volatility $\sigma$ so that a derivative can be taken. First take successive derivatives of $K(s)$ at zero in order to recover the general form for the $n^{th}$ order annualized cumulant,

$$c_n = (n - 1)! \lambda^n \nu$$  

(22)

Here, as in a diffusion setup, it is preferable to use volatility $\sigma$ instead of the horizon dependent standard deviation STD. For all Lévy processes the
quantities are related through

\[
\text{STD}(t) = (\text{Ex}_t^2)^{1/2} = (c_2t)^{1/2} = \sigma t^{1/2}
\]  \hspace{1cm} (23)

In the same way, it is preferable to focus on annualized skewness, \( \varsigma \), which is a horizon independent "rate" and relates to \( t \)-year skewness, \( \text{SKEW}(t) \), by

\[
\text{SKEW}(t) = \frac{\text{Ex}_t^3}{(\text{Ex}_t^2)^{3/2}} = \frac{c_3t}{(c_2t)^{3/2}} = \varsigma t^{-1/2}
\]  \hspace{1cm} (24)

where \( \varsigma = \frac{c_3}{c_2^{3/2}} \). Observe that while \( \text{STD}(t) \) is an increasing function of \( t \), from the central limit theorem \( \text{SKEW}(t) \) will tend to zero as \( t \) increases.

From (22) we then have the following system connecting \( \lambda \) and \( \nu \) to skewness \( \varsigma \) for the special case of a gamma process:

\[
c_2 = \sigma^2 = \lambda^2 \nu \hspace{1cm} \Rightarrow \hspace{1cm} \lambda = \frac{\sigma \varsigma}{2} \\
c_3 = 2\lambda^3 \nu \hspace{1cm} \nu = \frac{4}{\varsigma^3}
\]  \hspace{1cm} (25)

I can actually re-write the risk free rate (20), in the gamma economy, using moments as follows:

\[
\hat{r} = \delta + \vartheta \mu - \frac{2\sigma}{\varsigma} \vartheta + \frac{4}{\varsigma^2} \ln(1 + \frac{1}{2} \sigma \varsigma \vartheta) \\
= \delta + \vartheta \mu - \frac{2\sigma}{\varsigma} \vartheta - \frac{4}{\varsigma^2} \ln \varphi
\]  \hspace{1cm} (26)

where \( \varphi \) in (21) can be written as a function of moments,

\[
\varphi = (1 + \frac{1}{2} \sigma \varsigma \vartheta)^{-1}
\]  \hspace{1cm} (27)
It is then straightforward to see that for the case of the gamma process the volatility sensitivity equals

\[ \frac{\partial r_f}{\partial \sigma} = -\varphi \sigma \vartheta^2 < 0 \]

and \( \varphi \) is indeed the volatility sensitivity ratio of this economy \( \varphi = -\frac{\partial r_f/\partial \sigma}{\sigma \vartheta^2} \) (Definition 1).

As we saw already (theorem 1), when volatility is increased, agents who fear the impact of negative shocks are always saving more and risk free rates decrease. But furthermore in an economy driven by a catastrophic gamma associated with negative jumps \( (\lambda < 0) \), the \( \varphi \) factor is greater than one and the economy more sensitive; each unit of increase in catastrophic volatility has a magnified effect on the risk free rates. On the contrary, when the economy is booming, the sensitivity ratio is less than one, and interest rates are less sensitive to changes in volatility.

The next lemma is the specialization of theorem (1) for the gamma economy and supports the central assertion of the paper; i.e. that \( \varphi \) is ultimately related to hedging.

**Lemma 2** The risk aversion sensitivity of the risk free rate for the gamma
The economy equals

\[ \frac{\partial r^f}{\partial \vartheta} = \mu - \varphi \sigma^2 \vartheta \]

which is actually still valid for the Gaussian economy since \( \varphi = 1 \). Recall from the discussion related to theorem (1) that the positive \( \mu \) component of the risk aversion sensitivity of interest rates is due to the fact that risk aversion is inversely related to the elasticity of intertemporal substitution and thus an increased risk aversion leads to a lower elasticity; less elastic agents need a higher rate to induce them to postpone consumption. The component related to hedging risk across states is the negative \( -\varphi \sigma^2 \vartheta \). So unhedgeable catastrophic risk scales up this hedging related adjustment. On the other hand, an illiquid but booming market, \( \varphi < 1 \), deflates the risk related component; agents are less easily induced to save instead of investing on the booming equity.

If \( \varphi \) is to be a natural measure of the inability to hedge it should obviously grow with negative skewness. This is clear since:

**Lemma 3**  *The volatility sensitivity of the gamma process decreases with its skewness,*
\[
\frac{\partial \varphi}{\partial \varsigma} = -\frac{1}{2} \varphi^2 \sigma \vartheta < 0
\]

Agents who face a catastrophic risk should be more worried when such a risk becomes bigger. On the other hand, participation in a booming uncertainty makes them less sensitive to uncertainty itself. In a sense, prudent agents facing uncertainty resulting from positive jumps are bolder. This is an entirely new intuition with no analogue concept in the case of Gaussian risks:

**Lemma 4** In the case of a catastrophic gamma process an increased volatility results in increased sensitivity. On the contrary, when risk is booming an increased volatility lowers \( \varphi \).

**Proof:** Formally a gamma risk is catastrophic (booming) when the skewness is negative (positive). The lemma is straightforward since:

\[
\frac{\partial \varphi}{\partial \sigma} = -\frac{1}{2} \varphi^2 \varsigma \vartheta
\]

From the previous lemmas, a plot of \( \varphi \) against the jump induced skewness,
will be positive, strictly monotonically decreasing, and, crossing the purely
diffusive unit level at zero (see figure 2).

A **Gamma risk is scaled in the risk neural measure.**

In (12) and (13) the general relation between the risk neutral and actual Lévy
measures for CRRA economies was developed. It is well known that when
driven by a Brownian motion (10) the $x_t$ process has the same volatility, $\sigma$,
under both measures. By using (13), the risk neutral cumulant function,
$K^*(s)$, equals

$$K(s - \vartheta) - K(-\vartheta) = \frac{1}{2}\sigma^2(s - \vartheta)^2 - \frac{1}{2}\sigma^2\vartheta^2 = -s\sigma^2\vartheta + \frac{1}{2}\sigma^2s^2$$

which corresponds to a diffusion with the same volatility as in the historical
measure. More specifically in this case we get the usual

$$x_t = -\sigma^2\vartheta t + \sigma B^*_t$$

where $B^*_t$ is a standard Brownian motion under $Q$.

The risk neutral cumulants of a gamma process are **scaled** versions of its
respective empirical cumulants where the **illiquidity** of the gamma process,
$\varphi$, is the scaling factor,
**Theorem 2** Under the risk neutral distribution, gamma risks are scaled by their sensitivity ratio $\varphi$

**Proof:** From (12), under the risk neutral measure the risk neutral density of the Lévy process is a member of the exponential family of the original density. Thus the risk neutral cumulant function, $K^*(s)$, of the gamma process is given by (13):

$$
K(s - \vartheta) - K(-\vartheta) = -\lambda \nu (s - \vartheta) - \nu \ln(1 - \lambda (s - \vartheta)) - \lambda \nu \vartheta + \nu \ln(1 + \lambda \vartheta)
$$

$$
= -\lambda \nu s - \nu \ln \left( \frac{1 - \lambda (s - \vartheta)}{1 + \lambda \vartheta} \right)
$$

$$
= -\lambda \nu s - \nu \ln (1 - \varphi \lambda s)
$$

where $\varphi$ is given by (21). This is the cumulant function of a gamma process with a scale parameter $\lambda^* = \varphi \lambda$,

$$
x_t = -\lambda \nu t + \varphi \lambda \gamma^*_{\nu t} = -(1 - \varphi) \lambda \nu t + \varphi (\lambda \gamma^*_{\nu t} - \lambda \nu t)
$$

(29)

where $\gamma^*_{\nu t}$ is a gamma process under Q. $\blacksquare$

Risk neutral cumulants of any order $n \geq 2$ are directly recovered from the general equation (22) properly accounting for the correct risk-neutral scale.
\[
c_n^* = (n-1)! (\varphi \lambda)^n \nu = \varphi^n c_n
\]

Risk-neutral variance (the second cumulant) is scaled by \( \varphi^2 \), while, the third centralized moment is scaled by \( \varphi^3 \). In the diffusive case \( \varphi = 1 \), and thus the theorem is trivially valid for Brownian processes, since it only re-states that risk neutral and actual diffusive volatilities remain the same.

### B Inability to hedge inflates volatility.

Volatilities implied from Black-Scholes option pricing are well known to be significantly higher than actual volatilities computed over the option’s horizon. This phenomenon persists even for enhanced diffusive models such as the ones that allow for jumps in the price process and/or stochastic volatility. For example Bates (2000) finds that, for the stochastic volatility model to fit option prices, implausible values for the parameters of the volatility process would be needed; i.e. extremely high volatility of volatility.

Against this background, it is significant that unhedgeable gamma volatil-
ity in (29) is scaled:

\[ \text{gamma risk: } \sigma^* = \varphi \sigma \]  

(31)

as opposed to the invariance of Brownian risks (28) under the change of
measure.

For a catastrophic gamma risk, the increased volatility sensitivity associ-
ated with this risk, captured by \( \varphi > 1 \), inflates all the risk neutral cumulants.
In the case of a booming risk instead, the decreased volatility sensitivity,
\( \varphi < 1 \), deflates the risk. In a sense, agents who face a catastrophic gamma
risk (\( \varphi > 1 \)) are unable to completely hedge it, and, inflate risk neutral
volatility. On the contrary, when jumps are positive the scaling factor \( \varphi \)
becomes less than unit. Faced with a booming risk agents price assets in a
risk neutral world with lower than actual volatility.

Given difficulties to explain asset prices in a rational framework, a branch
of the asset pricing literature now proposes deviations from the neoclassical
framework where agents compute prices in a behavioral context. Relation
(31) is also interesting because it provides a link between a rational concern
for liquidity with the behavioral concept of over-confidence. Over-confidence
is modeled as an irrational tightening of the true underlying risk; that is a scaling with a scale parameter which is less than unit. I have just shown that such a tightening connects the underlying (true) to the risk neutral distribution when rational agents price assets exposed to an explosive non-diffusive growth. So in this sense, when faced with non-diffusive booming risks, rational agents exhibit behavior that may be difficult to distinguish from over-confidence.

III Hedging multiple risks

In the case of an economy driven by a single risk we can safely associate the total sensitivity ratio with this risk, since the entire volatility is generated by this single uncertainty. It is nevertheless important, to understand the additive properties of $\phi$; i.e. understand the decomposition of the total sensitivity to sensitivities in individual risks.

For example, it may be advantageous to combine the infinite arrival of small jumps of the gamma process with a continuous path Brownian motion to generate a more suitable process where the gamma is capturing the illiquid
return component in price dynamics. Then in such a mixture with a diffusive risk, the $R^2$ of the unhedgeable gamma component will allude to a natural measure of illiquidity in the market.

Another example where the study of mixed risks emerges is in time changed processes. The mono-directional jumps of the pure gamma process are unlikely to capture the entire uncertainty in stock returns. The main deficiency is that jump activity is of only one type, either catastrophic or booming. On the other hand, exactly because of the mono-directionalility of its jumps, the gamma is a natural candidate in modeling passage of business time. When a gamma process ”changes” the time of a Brownian motion the resulting process is called Variance gamma (VG). Madan, Carr and Chang (1998) show that the VG process provides the much needed generalization to the gamma since it incorporates both catastrophic and booming components. More specifically the VG process is the difference between two pure gamma.

To study such richer risk structures, I generalize (3) and model growth as

$$c_t = \mu t + x_t = \mu t + \sum_i x_t^i$$  \hspace{1cm} (32)
where total risk $x_t$ is now composed of a set of independent Lévy risks $\sum_i x^i_t$ and, as in the single risk case, $E x^i_t = 0$ for all $i$.

Since now total volatility is decomposed in the volatilities of the individual processes we first have to clearly define what happens when volatility is increased; i.e. how the total derivative with respect to volatility is taken. By a change in total risk in (32) I mean a change in $\sigma$ where all risk components are scaled proportionately. Formally the total derivative will be taken with respect to a change in $\sigma$ that preserves the $R^2$ of individual risks. In this case, one has to differentiate the economy wide $\varphi$ -by Definition 1- from the sensitivity ratio, $\varphi_i$, associated with the $i_{th}$ process.

The key in proving the next theorem is the fact that when the processes are independent the total cumulant function is additive,

$$K(s) = \sum_i K_i(s)$$

where $K_i(s) = \frac{1}{i} \ln E e^{sx^i_t}$ is the cumulant function of the $i^{th}$ process $x^i_t$. It is then straightforward to see that the total precautionary savings effect is additive,

$$K(-\vartheta) = \sum_i K_i(-\vartheta)$$

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and the risk free rate is given by

\[ r^f = \delta + \vartheta \mu - \sum_i K_i(-\vartheta) \]  

(35)

Thus the total volatility sensitivity will equal

\[ \frac{dr^f}{d\sigma} = -\sum_i \frac{\partial K_i(-\vartheta)}{\partial \sigma_i} \frac{d\sigma_i}{d\sigma} \]  

(36)

From the total variance decomposition,\(^7\) \( R_i = \frac{\sigma_i}{\sigma} \), a change \( d\sigma \) that preserves individual \( R_i^2 \), is such that

\[ d\sigma_i = R_i d\sigma \]  

(37)

Expanding the total derivative we get

\[ \frac{dr^f}{d\sigma} = -\sum_i \frac{\partial K_i(-\vartheta)}{\partial \sigma_i} \frac{d\sigma_i}{d\sigma} = -\sum_i R_i \frac{\partial K_i(-\vartheta)}{\partial \sigma_i} \]

Finally dividing by the diffusive sensitivity, \(-\sigma \vartheta^2\), we get

\[ \varphi = \frac{1}{\sigma \vartheta^2} \sum_i R_i \frac{\partial K_i(-\vartheta)}{\partial \sigma_i} = \sum_i R_i^2 \left( \frac{1}{\sigma_i \vartheta^2} \frac{\partial K_i(-\vartheta)}{\partial \sigma_i} \right) \]

where the term inside the parenthesis is indeed equal to the sensitivity ratio associated with the \( i_{th} \) risk,

\[ \varphi_i = \frac{1}{\sigma_i \vartheta^2} \frac{\partial K_i(-\vartheta)}{\partial \sigma_i} \]

I have thus proved,

\(^7\)Since in the case of Lévy processes variance increases linearly with time, \( R_i^2 \) does not depend on the horizon.
Theorem 3 The total inability, $\varphi$, to hedge a composite risk is equal to the $R^2$-weighted sum of the sensitivity ratios of the individual processes, $\varphi = \sum_i R^2_i \varphi_i$.

Thus the total ratio is equal to a properly weighted convex combination of all sensitivity ratios, the weights, $R^2_i$, are equal to the proportions of total volatility captured by each risk, $R^2_i = \frac{\sigma^2}{\sigma^2}$.

A Examples of mixed risks

The gamma-Brownian mixture. In this case, I assume an economy that faces a combination of a gamma risk $x^o$, and an independent Brownian noise $\varepsilon_t = \sigma \varepsilon B_t$,

$$x_t = \lambda \gamma_{vt} - \lambda \nu t + \sigma \varepsilon B_t + \varepsilon_t$$

where $B_t$ is a standard Brownian motion. This process is clearly superior to (18) and (10) since it nests both continuous and pure jump components. Since, by definition, there is one unit of $\varphi$ associated with the Brownian motion, the inability $\varphi$ to hedge (38) will be equal to,

$$\varphi = R^2_o \varphi_o + 1 - R^2_o$$

(39)
where $R^2_o$ of the total variance is explained by the non-diffusive risk. The $\varphi_o$ associated with the gamma risk is given by (21), so that

$$\varphi = R^2_o (1 + \lambda \vartheta)^{-1} + 1 - R^2_o$$  \hspace{1cm} (40)

**Time changed Brownian motion.** The VG process is a time changed Brownian motion where a pure gamma $\tau_t = \lambda \gamma_{\nu t}$ is used to measure the passage from real $t$ to trading time. It has been shown (e.g. Madan, Carr and Chang, 1998) that the VG is equal to the sum of a booming pure gamma process, $\lambda_b \gamma_{bt}^{b}$, to a catastrophic process, $-\lambda_c \gamma_{ct}^{c}$,  

$$x_t = \lambda_b \gamma_{bt}^{b} - \lambda_c \gamma_{ct}^{c} \quad \text{where} \quad \lambda_b, \lambda_c > 0$$  \hspace{1cm} (41)

where both $\gamma^b$ and $\gamma^c$ processes share the same $\nu$ parameter.

Applying Theorem (3) directly we get the total inability to hedge a VG process,

$$\varphi = R^2_b \times \frac{1}{1 + \lambda_b \vartheta} + R^2_c \times \frac{1}{1 - \lambda_c \vartheta}$$  \hspace{1cm} (42)

where

$$R^2_b = \frac{\sigma^2_b}{\sigma^2_b + \sigma^2_c} = \frac{\lambda^2_b}{\lambda^2_b + \lambda^2_c}$$  \hspace{1cm} (43)

is the $R^2$ of the booming component, and $R^2_b + R^2_c = 1$. Observe that for the
booming component,
\[ \varphi_b = \frac{1}{1 + \lambda_b \vartheta} < 1 \]  
while for the catastrophic,
\[ \varphi_c = \frac{1}{1 - \lambda_c \vartheta} > 1 \]  
Finally observe that under the risk neutral measure (31) the booming risk is *deflated*,
\[ \sigma_b^* = \varphi_b \times \sigma_b \]  
while the catastrophic risk is *inflated*,
\[ \sigma_c^* = \varphi_c \times \sigma_c \]  
From
\[ \lambda_b^* = \varphi_b \lambda_b < \lambda_b \]  
we see that the risk neutral Lévy measure of the booming gamma
\[ \rho_b^*(dx) = \frac{\nu}{x} e^{-x/\lambda_b^*} dx \quad \text{for } x>0 \]  
implies that risk neutral positive jumps will always arrive at a lower rate
\[ \rho_b^*(dx) < \rho_b(dx) \]
The opposite happens to risk neutral negative jumps that arrive at a higher than actual rate. This implies that risk neutral skewness is more negative than actual.

B Correcting the Drift

In an uncertain environment, agents will always lower risk neutral drift to insure themselves against bad future states. In the traditional diffusion setup this drift lowering has come to be recognized as a direct result of risk aversion.

In a more general Lévy framework, where risks may contain pure jump unhedgeable components, this drift lowering will not only capture risk aversion but will also compensate for the inability to completely hedge. As is shown next, the sensitivity ratio $\varphi$ allows us to differentiate between the purely diffusive effect from the inability to hedge that is only present when the pure jump component exists.

In the Brownian case (28) risk correction is attained by a displacement of the drift to $\mu^* = \mu - \sigma^2 \vartheta$. Essentially, this is a concept of certainty equivalence. For, rational agents discount the expected value of a risky

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growth by $\sigma^2 \vartheta$ to arrive at an equivalent deterministic value. Intuitively, this discount depends on the risk aversion times variance product, since the more risk averse agents are the less happy they are about economic risk. The curvature of their utility and the variance of the risk determine the magnitude of the Jensen effect. The natural question is what is the analogous displacement that agents apply when faced with gamma risks.

**Theorem 4** When Brownian and gamma components are mixed (38), the diffusive drift discount $\sigma^2 \vartheta$ is scaled by the total inability to hedge,

$$\mu^* = \mu - \varphi \sigma^2 \vartheta$$

(51)

**Proof:** I have shown in (29) that under Q a zero drift gamma process (18) is drifted lower as in $x_t = \varphi \lambda \gamma^*_t - \lambda \nu t$. This can also be written as

$$x_t = -(1 - \varphi)\lambda \nu t + \varphi(\lambda \gamma^*_t - \lambda \nu t)$$

where, the negative sign of the drift correction $-(1 - \varphi)\lambda \nu$ can be seen from the fact that

$$\varphi > 1 \quad \text{iff} \quad \lambda < 0$$

(52)

From $\varphi = (1 + \lambda \vartheta)^{-1}$, and $\sigma^2 = \lambda^2 \nu$ we can see that risk neutral drift is lowered by

$$-(1 - \varphi)\lambda \nu = -\varphi \sigma^2 \vartheta < 0$$

(53)
Now let us denote by $\Gamma$ and by $N$ the set of gamma and Brownian processes respectively. Then

$$c_t = \mu t + \sum_{i} x_i^t = \mu t + \sum_{i \in \Gamma} x_i^t + \sum_{j \in N} x_j^t$$  \hspace{1cm} (54)

Under the risk neutral measure, the total drift lowering of growth will be equal to

$$\sum_{i \in \Gamma} \phi_i \sigma_i^2 \vartheta + \sum_{j \in N} \sigma_j^2 \vartheta = \sum_{i} \phi_i \sigma_i^2 \vartheta = \sum_{i} \phi_i R_i^2 \sigma^2 \vartheta = \phi \sigma^2 \vartheta$$  \hspace{1cm} (55)

since by Theorem (3) total $\phi = \sum_i \phi_i R_i^2$. \hfill \blacksquare

That is, the presence of unhedgeable risks induces a discount which is, conveniently, equal to the discount that would apply if the total risk was diffusive, $\sigma^2 \vartheta$, scaled by the $\phi$ factor. When negatively skewed risks ($\phi_i > 1$) dominate, the total $\phi = \sum_i \phi_i R_i^2$ is greater than unit and thus agents discount the uncertain growth more sharply.

IV Conclusion

The purely diffusive case is extended to a general theory relating volatility sensitivity to the type of the underlying Lévy process. In Theorem 1 the
exact measure of how precautionary savings lower the risk free rate for all Lévy processes is recovered. The sensitivity ratio $\varphi$ is defined in order to disentangle pure aversion to risk -which is present even in the pure diffusion case- from inability to hedge -that is only present when the pure jump component exists. The intuition of the definition is that the differential intensity of the precautionary saving motive when agents are exposed to risk generated by a Lévy process should capture their inability to hedge its pure jump component.

Unlike a Brownian motion, a pure gamma always has a non-zero drift. This can cause a confusion when in manipulating the parameters of the gamma to calibrate higher moments one is also changing the drift. The novelty of the paper is that in order to effectively deal with this issue, and be able to measure the true volatility sensitivity of a process, the drift of the original gamma process is removed. Thus in this paper, besides volatility $\sigma$, the other natural parameter of the gamma becomes skewness $\varsigma$ instead of the usual drift $\rho$.

I show that, unlike Brownian risk that remains invariant under $Q$, gamma risk is scaled by its sensitivity $\varphi$. It is shown that the ”inability to hedge”
factor $\varphi$ depends on both volatility and skewness. The $\varphi$ ratio is also shown to scale the risk-related lowering of the drift of the gamma.

In the last section, the inability to hedge composite risks is addressed by Theorem 3. This theorem allows the direct calculation of the sensitivity of a time changed Brownian motion, the VG process. The VG process provides a much needed generalization to the gamma which allows only jump activity of a single direction. When the gamma risk is booming or positive $\varphi$ is less than one. When the gamma risk is catastrophic $\varphi > 1$. Then under Q for a VG process the booming component will be deflated while the catastrophic component inflated.
V Appendix

A The risk neutral measure $Q$

From (5) CRRA agents stochastically discount cash flows at a rate $\xi_t = \delta t + \vartheta \mu t + \vartheta x_t$. The measure $Q$ is a member of the exponential family of the real measure $P$. In this change of measure the negative risk aversion, $-\vartheta$ becomes, what is in Statistics known as, the tilting vector. The Radon-Nikodym derivative equals

$$\left( \frac{dQ}{dP} \right)_t = e^{-\vartheta x_t - tK(-\vartheta)}$$

where $K(s)$ denotes the cumulant function of $x_t$. In other words, $K(s)$ is the annualized logarithm of the conditional moment generating function $M(s)$:

$$tK(s) = \ln M(s) = \ln E e^{sx_t}$$

To derive the risk neutral measure, start with the way agents price future cash flows by discounting,

$$S_0 = E e^{-\xi_t} S_t = e^{-\delta t - \vartheta \mu t} E e^{-\vartheta x_t} S_t$$
and, apply it to a riskless t-year bond,

\[ e^{-rt} = e^{-\delta t - \vartheta \mu t} E e^{-\vartheta x_t} t K(-\vartheta) \]

Dividing the two equations we get

\[ S_o e^{rt} = E e^{-\vartheta x_t - t K(-\vartheta)} S_t \Rightarrow S_o = E^s e^{-rt} S_t \]

which is the usual equation saying that prices equal risk-neutrally expected discounted future cash flows.

Finally the cumulant function under the risk neutral measure satisfies

\[ t K^*(s) = \ln E^s e^{sx_t} = \ln E e^{-\vartheta x_t - t K(-\vartheta)} e^{sx_t} = t K(s - \vartheta) - t K(-\vartheta) \]
References


Figure 1. Unhedgeable risks and the cumulant function. Plot of cumulant generating function for a Lévy process $x_t$ for various values of skewness, while keeping variance constant. For the (18) pure gamma process $K(s) = -\nu \ln(1 - \lambda s) - \lambda \nu s$. For a zero drift process, the slope at the origin is always zero. Dotted line depicts the quadratic function for a Brownian motion; $K(s) = \frac{1}{2}\sigma^2 s^2$. Remember that $r^f = \delta + \vartheta \mu - K(-\vartheta)$. The convexity of $K(s)$ clearly implies precautionary savings; i.e. $K(-\vartheta) > 0$. For a Lévy
process \( x_t \) the negative derivative at minus risk aversion, \( K'(-\vartheta) < 0 \), is the negative risk neutral drift. When jumps are negative, \( K(s) \) becomes more convex for negative values and less convex for positive values, since negative skewness implies a negative third derivative and thus a decreasing second derivative which determines convexity -second derivatives at zero are fixed by volatility, \( K''(0) = \sigma^2 \). For all plots \( \sigma = 20\% \). Thick solid curve depicts gamma \( K(s) \) for annualized skewness \( \varsigma = -1 \). Thin solid curve plots the gamma with \( \varsigma = -2 \).
Figure 2. The sensitivity depends on skewness. Plot of $\varphi$ as a function of skewness for various $\vartheta$ values in gamma economies. When gamma risk is catastrophic (negative jumps), $\varphi > 1$. The ratio decreases with skewness, $\frac{\partial \varphi}{\partial \varsigma} < 0$, and is a convex function of skewness, $\frac{\partial^2 \varphi}{\partial \varsigma^2} > 0$. For all plots $\sigma = 20\%$. 
Figure 3. Risk free rate and unhedgeable risk. The effect of non-diffusive risk on $r^f$ for various degrees of $\vartheta$. A catastrophic non-diffusive gamma process lowers $r^f$. The Gaussian rates are represented by the middle curve. When the economy is booming (positive unhedgeable jumps) agents prefer to participate in equity and interest rates go up. For all curves here a time preference parameter $\delta$ equal to 1% was used.