Market Microstructure Invariants

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Motivation

It is important for asset managers to understand

- The level of transaction costs:
  - market impact (increasing cost per share traded),
  - bid-ask spread (fixed cost pre share traded);

- How transactions costs vary cross-sectionally across stocks as level of trading activity varies.
Motivation

Understanding level of transaction costs helps answer some important questions:

- What percentage of “alpha” is lost due to transaction costs?
- How much money can be allocated to a seemingly profitable strategy before it becomes non-economical due to high transaction costs?
Motivation

Understanding **cross-sectional variation** in **transaction costs** helps answer the following questions:

- Is it reasonable to restrict the rate of trading to a fixed percentage of trading volume, say 1% of daily volume for all stocks, or should the maximum percentage of average daily volume vary across stocks?

- If one broker executes orders for small stocks and another broker executes orders for large stocks, how can we compare their performance?
Overview

Our goal is to explain how trade size, trade frequency, market impact and bid-ask spread vary across stocks with different trading activity.

- We develop a model of market microstructure invariants that generates predictions concerning cross-sectional variations of these variables.
- These predictions are tested using a data set of portfolio transitions and find a strong support in the data.
- The model implies a simple formula for market impact and bid-ask spread as functions of observable dollar trading volume and volatility, which provides answers for above questions.
A Framework

When portfolio managers trade stocks, they can be thought of as playing trading games. Since managers trade many different stocks, we can think of them as playing many different trading games simultaneously, a different game for each stock.

The intuition behind a trading game was first described by Jack Treynor (1971). In that game informed traders, noise traders and market makers traded with each other.
Reduced Form Approach

We assume that trades arrive according to a compound Poisson process with trade arrival rate $\gamma$ and trade size being a random variable $\tilde{Q}$.

$\tilde{Q}$ and $\gamma$ vary across stocks.
Bets

We think of trades as bets whose size is measured by dollar standard deviation over time.

Bet size over a calendar day:

\[ \tilde{B} = P \cdot \tilde{Q} \cdot \sigma \]

Bet size increases as a square root with time.
Bet Frequency

Bets arrive in the market with an assumed frequency.

Bet frequency per calendar day:

\[ \gamma \]

Bet frequency increases proportionally with time.
Trading Activity

Stocks differ in their “Trading Activity” $W$, or a measure of gross risk transfer, defined as dollar volume adjusted for volatility $\sigma$:

$$W = V \cdot P \cdot \sigma = \gamma \cdot \mathbb{E}\{|\tilde{B}|\}.$$
Theoretical Irrelevance Principles

Modigliani-Miller Irrelevance: The trading game involving a financial security issued by a firm is independent of its capital structure:

- Stock Split Irrelevance,
- Leverage Irrelevance.

Time-Clock Irrelevance: The trading game is independent of the speed at which the time clock ticks.
Trading Game Invariance

Irrelevances imply trading game invariant $\tilde{I}$:

$$\tilde{I} = \frac{P \cdot \tilde{Q} \cdot \sigma}{\gamma^{1/2}} = \frac{\tilde{B}}{\gamma^{1/2}} = \text{const.}$$

Bet frequency $\gamma$ increases twice as fast as bet size $\tilde{B}$, as trading speeds up. Both are not affected by stock splits and changes in leverage. Trading game takes place in transaction time.
Bets and Trading Activity

**Bet frequency** increases *twice as fast as bet size*, as trading speeds up, and trading activity $W$ is equal

$$W = V \cdot P \cdot \sigma = \gamma \cdot \mathbb{E}\{|\tilde{B}|\},$$

**Implications:** bet frequency $\gamma \sim W^{2/3}$ and bet size $\tilde{B} \sim W^{1/3}$!
Price Impact and Spread

Our invariance principle has implications for **price impact** and **bid-ask spread**. We make assumptions which are consistent with many models, including adverse selection and inventory models:

- Price volatility results from linear price impact of trades.
- Spread is proportional to standard deviation per trading day.

**Implications:** price impact $\lambda V/\sigma P \sim W^{1/3}$ and spread $\kappa \sim W^{-1/3}$!
Empirical Hypothesis: Market Microstructure Invariants

For different securities and the same securities at different times:

- **Trading Game Invariance:** The distributions of market microstructure invariants $\tilde{I} = \frac{\tilde{B}}{\gamma^{1/2}}$ are the same.

- **Market Impact Invariance:** Linear price impact of trades generates the same constant fraction of returns variance.

- **Bid-Ask Spread Invariance:** Bid-ask spread cost of a bet is the same constant fraction of impact costs.
Games Across Stocks

Stocks are different in terms of their trading activity: dollar trading volume, volatility etc. Trading games look different across stocks only at first sight!

Our intuition is that trading games are the same across stocks, except for the length of time over which these games are played or the speed with which they are played.
Games Across Stocks

Only the speed with which time passes varies, when trading activity varies:

- For active stocks (high trading volume and high volatility), trading games are played at a fast pace, i.e. the length of trading day is small.

- For inactive stocks (low trading volume and low volatility), trading games are played at a slow pace, i.e. the length of trading day is large.

The length of a trading day is related to market efficiency. The shorter is the trading day, the more efficient is the market.
A Benchmark Stock

**Benchmark Stock** - daily volatility $\sigma = 200$ bps, price $P^* = $40, volume $V^* = 1$ million shares. Trades over a calendar day:

Arrival Rate $\gamma^* = 4$

Avg. Order Size $\bar{Q}^*$ as fraction of $V^* = 1/4$

Market Impact of $1/4 \ V^* = 200$ bps / $4^{1/2} = 100$ bps

Spread $= k$ bps
Market Microstructure Invariants - Intuition

**Benchmark Stock with Volume** \(V^*\)  
\((\gamma^*, \tilde{Q}^*)\)

**Stock with Volume** \(V = 8 \cdot V^*\)  
\((\gamma = \gamma^* \cdot 4, \tilde{Q} = \tilde{Q}^* \cdot 2)\)

Avg. Order Size \(\tilde{Q}^*\) as fraction of \(V^*\)  
\(= \frac{1}{4}\)

Market Impact of \(1/4\) \(V^*\)  
\(= 200\) bps / \(4^{1/2} = 100\) bps

**Spread**  
\(= k\) bps

Market Impact of \(1/16\) \(V\)  
\(= 200\) bps / \((4 \cdot 8^{2/3})^{1/2} = 50\) bps

Avg. Order Size \(\tilde{Q}\) as fraction of \(V\)  
\(= \frac{1}{16} = \frac{1}{4} \cdot 8^{-2/3}\)

Market Impact of \(1/4\) \(V\)  
\(= 4 \cdot 50\) bps = \(100\) bps \(\cdot 8^{1/3}\)

**Spread**  
\(= k\) bps \(\cdot 8^{-1/3}\)
If trading activity $W$ increases by one percent, some algebra implies the following cross-sectional predictions:

- **Trade Size**, as a percentage of average daily volume, decreases by $\frac{2}{3}$ of one percent;
- **Market impact** of trading $X$ percent of average daily volume increases by $\frac{1}{3}$ of one percent;
- **Bid-ask spread** decreases by $\frac{1}{3}$ of one percent.
Market Microstructure Invariants - Prediction

Math

The Model of Market Microstructure Invariants implies:

- **Market Impact**: \( \lambda_{TG} = const \cdot W^{1/3} \cdot \frac{\sigma P}{V} \),
- **Bid-Ask Spread**: \( k_{TG} = const \cdot W^{-1/3} \cdot \sigma P \),
- **Order Size**: \( \frac{\bar{Q}_{TG}}{V} = const \cdot |\bar{B}_H| \cdot W^{-2/3} \).
- **Length of Trading Day**: \( H = const \cdot W^{-2/3} \)

where \( W = V \cdot P \cdot \sigma \) is the trading activity.
Alternative Theories

We consider two alternative theories:

1. Naive alternative **Model of “Invariant Bet Frequency”** based on intuition that as trading activity increases, the size of bets increases proportionally, but their arrival rate remains constant.

2. Naive alternative **Model of “Invariant Bet Size** based on the intuition that as trading activity increases, the size of bets remains the same, but their arrival rate increases proportionally.
Model of Invariant Bet Frequency

Model of **Invariant Bet Frequency** assumes that all variation in trading activity $W$ is explained entirely by variation in **bet size**.

As trading activity varies across stocks,

- Bet size $\tilde{B}$ **varies** proportionally.
- Bet frequency $\gamma$ remains **constant**.
Invariant Bet Frequency - Intuition

Benchmark Stock with Volume $V^*$
$(\gamma^*, \tilde{Q}^*)$

Stock with Volume $V = 8 \cdot V^*$
$(\gamma = \gamma^*, \tilde{Q} = \tilde{Q}^* \cdot 8)$

Avg. Order Size $\tilde{Q}^*$ as fraction of $V^* = 1/4$

Market Impact of $1/4 V^*$
$= 200 \text{ bps} / 4^{1/2} = 100 \text{ bps}$

Spread
$= k \text{ bps}$

Avg. Order Size $\tilde{Q}$ as fraction of $V = 1/4$

Market Impact of $1/4 V$
$= 200 \text{ bps} / 4^{1/2} = 100 \text{ bps}$

Spread
$= k \text{ bps}$
Invariant Bet Frequency - Predictions

If trading activity increases by one percent, then some math implies the following cross-sectional predictions:

- **Trade Size**, as a fraction of average daily volume, is constant because order size increases proportionally with average daily volume;

- **Market impact** of trading $X$ percent of average daily volume is constant;

- **Bid-ask spread** is constant.

**Intuition:** There is the same number of independent (but larger) trades per day, trading volume and order imbalances increase at the same rate; market depth does not change.
Invariant Bet Frequency - Comment

We believe that the model of Invariant Bet Frequency is the “default model” that implicitly but incorrectly guides the intuition of many asset managers.

- Model justifies trading say no more than 1% of average daily volume for all stocks, regardless of level of trading activity.

- Model justifies imputing same number of basis points in transactions costs for individual stocks in a basket with both active and inactive stocks, where size of trades are proportional to average daily volume.
Invariant Bet Frequency - Prediction Math

The Model of Invariant Bet Frequency implies:

- **Market Impact**: $\lambda_\gamma = \text{const} \cdot W^0 \cdot \frac{\sigma P}{V}$,
- **Bid-Ask Spread**: $k_\gamma = \text{const} \cdot W^0 \cdot \sigma P$,
- **Order Size**: $\frac{|\tilde{Q}_\gamma|}{V} = \text{const} \cdot |\tilde{Z}| \cdot W^0$,
- **Length of Trading Day**: $H = 1 \cdot W^0$,

where $W = V \cdot P \cdot \sigma$ is the trading activity.
Model of Invariant Bet Size

Model of **Invariant Bet Size** assumes that all variation in trading activity is explained exclusively by variation in **bet frequency**.

As trading activity varies across stocks,

- Bet size $\tilde{B}$ remains **constant**.
- Bet frequency $\gamma$ **varies** proportionally.
Invariant Bet Size - Intuition

**Benchmark Stock with Volume** \( V^* \)

\[
(\gamma^*, \tilde{Q}^*)
\]

**Stock with Volume** \( V = 8 \cdot V^* \)

\[
(\gamma = \gamma^* \cdot 8, \tilde{Q} = \tilde{Q}^*)
\]

Avg. Order Size \( \tilde{Q}^* \) as fraction of \( V^* \)

\[
= 1/4
\]

Market Impact of 1/4 \( V^* \)

\[
= 200 \text{ bps} / 4^{1/2} = 100 \text{ bps}
\]

\[
\text{Spread} = k \text{ bps}
\]

Market Impact of 1/32 \( V \)

\[
= 200 \text{ bps} / 32^{1/2}
\]

Avg. Order Size \( \tilde{Q} \) as fraction of \( V \)

\[
= 1/32 = 1/4 \cdot 8^{-1}
\]

Market Impact of 1/4 \( V \)

\[
= 8 \cdot 200 \text{ bps} / 32^{1/2} \text{ bps} = 100 \text{ bps} \cdot 8^{1/2}
\]

\[
\text{Spread} = k \text{ bps} \cdot 8^{-1/2}
\]
Invariant Bet Size - Predictions

If trading activity increases by one percent, then some math implies the following cross-sectional predictions:

- **Trade Size**, as a fraction of average daily volume, decreases by one percent;
- **Market impact** of trading $X$ percent of average daily volume increase by $1/2$ of one percent;
- **Bid-ask spread** decreases by $1/2$ of one percent.

**Intuition:** Since there are more independent liquidity trades per day, trading volume increases twice as fast as order imbalances. Thus, market depth increases at half the rate as trading volume.
Invariant Bet Size - Prediction Math

The Model of Invariant Bet Size implies:

- **Market Impact**: \( \lambda_B = \text{const} \cdot W^{1/2} \cdot \frac{\sigma P}{V} \),
- **Bid-Ask Spread**: \( k_B = \text{const} \cdot W^{-1/2} \cdot \sigma P \),
- **Order Size**: \( \frac{|\tilde{Q}_B|}{V} = \text{const} \cdot |\tilde{B}| \cdot W^{-1} \),
- **Length of Trading Day**: \( H = 1 \cdot W^0 \),

where \( W = V \cdot P \cdot \sigma \) is the trading activity.
Identification

- Note that the level of market impact, the level of bid-ask spreads, and the average size of liquidity trades are not identified by the theory, but can be estimated from data.

- Note that the length of the trading day $H$ is not identified as well, but we cannot estimate it from data using our methodology either.
The empirical implications of the three proposed models are tested using a proprietary dataset of **portfolio transitions**.

- Portfolio transition occurs when an old (legacy) portfolio is replaced with a new (target) portfolio during replacement of fund management or changes in asset allocation.

- Our data includes 2,680+ portfolio transitions executed by a large vendor of portfolio transition services over the period from 2001 to 2005.

- Dataset reports executions of 400,000+ orders with average size of about 4% of ADV.
Portfolio Transitions and Trades

We use the data on transition orders to examine which model makes the most reasonable assumptions about how the size of trades varies with trading activity.
Tests for Orders Size - Design

All three models are nested into one specification that relates trading activity $W$ and the trade size $\tilde{Q}$, proxied by a transition order of $X$ shares, as a fraction of average daily volume $V$:

$$\ln \left[ \frac{X_i}{V_i} \right] = \bar{q} + a_0 \cdot \ln \left[ \frac{W_i}{W_*} \right] + \tilde{\epsilon}$$

The variables are scaled so that $e^{\bar{q}} \cdot 10^4$ is (assuming log-normal distribution) the median size of liquidity trade as a fraction of daily volume (in bps) for a benchmark stock with:

- daily standard deviation of 2%,
- price of $40$ per share,
- trading volume of 1 million shares per day,
- trading activity $W_* = 2\% \cdot 40 \cdot 1$ million.
Tests for Orders Size - Design

Three models differ only in their predictions about parameter $a_0$.

- **Model of Trading Game Invariance**: $a_0 = -2/3$.
- **Model of Invariant Bet Frequency**: $a_0 = 0$.
- **Model of Invariant Bet Size**: $a_0 = -1$.

We estimate the parameter $a_0$ to examine which of three models make the most reasonable assumptions.
## Tests for Orders Size - Results

<table>
<thead>
<tr>
<th></th>
<th>NYSE</th>
<th>NASDAQ</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>All</td>
<td>Buy</td>
</tr>
<tr>
<td>$\bar{q}$</td>
<td>$-5.67^{***}$</td>
<td>$-5.68^{***}$</td>
</tr>
<tr>
<td></td>
<td>(0.017)</td>
<td>(0.022)</td>
</tr>
<tr>
<td>$a_0$</td>
<td>$-0.63^{***}$</td>
<td>$-0.63^{***}$</td>
</tr>
<tr>
<td></td>
<td>(0.008)</td>
<td>(0.010)</td>
</tr>
</tbody>
</table>

- **Model of Trading Game Invariance:** $a_0 = -2/3$.
- **Model of Invariant Bet Frequency:** $a_0 = 0$.
- **Model of Invariant Bet Size:** $a_0 = -1$.

***is 1%-significance, **is 5%-significance, *is 10%-significance.
## Tests for Orders Size - F-Tests

<table>
<thead>
<tr>
<th></th>
<th>NYSE</th>
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<th></th>
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</thead>
<tbody>
<tr>
<td></td>
<td>All</td>
<td>Buy</td>
<td>Sell</td>
<td>Buy</td>
<td>Sell</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>F-test</strong></td>
<td>17.03</td>
<td>13.74</td>
<td>72.00</td>
<td>6.53</td>
<td>18.56</td>
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<tr>
<td><strong>p-val</strong></td>
<td>0.0000</td>
<td>0.0002</td>
<td>0.0000</td>
<td>0.0107</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

### Model of Trading Game Invariance: \( a_0 = -2/3 \)

<table>
<thead>
<tr>
<th></th>
<th>NYSE</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>All</td>
<td>Buy</td>
<td>Sell</td>
<td>Buy</td>
<td>Sell</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>F-test</strong></td>
<td>5664.91</td>
<td>3740.45</td>
<td>5667.60</td>
<td>1440.32</td>
<td>2427.51</td>
</tr>
<tr>
<td><strong>p-val</strong></td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

### Model of Invariant Bet Frequency: \( a_0 = 0 \)

<table>
<thead>
<tr>
<th></th>
<th>NYSE</th>
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</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>All</td>
<td>Buy</td>
<td>Sell</td>
<td>Buy</td>
<td>Sell</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td><strong>F-test</strong></td>
<td>1920.52</td>
<td>1306.11</td>
<td>2537.08</td>
<td>229.30</td>
<td>966.99</td>
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<tr>
<td><strong>p-val</strong></td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
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## Tests for Orders Size - $R^2$

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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>All</td>
<td>Buy</td>
<td>Sell</td>
<td>Buy</td>
</tr>
<tr>
<td>Three Parameters: $P$, $V$, $\sigma$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Adj. R^2$</td>
<td>0.3211</td>
<td>0.2614</td>
<td>0.2682</td>
<td>0.4382</td>
</tr>
<tr>
<td>One Parameter:</td>
<td>$W = P \cdot V \cdot \sigma$</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$Adj. R^2$</td>
<td>0.3188</td>
<td>0.2588</td>
<td>0.2643</td>
<td>0.4364</td>
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<tr>
<td>Model of Trading Game Invariance: $a_0 = -\frac{2}{3}$</td>
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<td></td>
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<td></td>
</tr>
<tr>
<td>$Adj. R^2$</td>
<td>0.3177</td>
<td>0.2577</td>
<td>0.2608</td>
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<tr>
<td>Model of Invariant Bet Frequency: $a_0 = 0$</td>
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<td></td>
</tr>
<tr>
<td>$Adj. R^2$</td>
<td>-0.0002</td>
<td>-0.0002</td>
<td>-0.0002</td>
<td>-0.0004</td>
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<tr>
<td>Model of Invariant Bet Size: $a_0 = -1$</td>
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<td></td>
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</tr>
<tr>
<td>$Adj. R^2$</td>
<td>0.2105</td>
<td>0.1683</td>
<td>0.1458</td>
<td>0.3669</td>
</tr>
</tbody>
</table>
Tests for Orders Size - Summary

Model of Trading Game Invariance assumes: An increase of one percent in trading activity $W$ leads to a decrease of $2/3$ of one percent in size of liquidity trade as a fraction of daily volume (for constant returns volatility).

Results: The estimates provide strong support for Model of Trading Game Invariance. The coefficient predicted to be $-2/3$ is estimated to be $-0.63$.

Discussion:

- The assumptions made in our model match the data economically.
- F-test rejects our model statistically because of small standard errors.
- Alternative models are rejected soundly with very large F-values.
- Estimating coefficients on $P$, $V$, $\sigma$ improves $R^2$ very little compared with imposing coefficient value of $-2/3$. 
Order Sizes Across Volume Groups

Do the data match models’ assumptions across 10 volume groups?

\[
\ln \left[ \frac{X_i}{V_{1,i}} \right] = \left[ \sum_{j=1}^{10} I_{j,i} q_j \right] + a_0 \cdot \ln \left[ \frac{W_i}{W_*} \right] + \tilde{\epsilon}
\]

- **Parameter** $a_0$ is restricted to values predicted by each model ($a_0 = -2/3$, $a_0 = 0$, or $a_0 = -1$).

- Indicator variable $I_{j,i}$ is one if $i$th order is in the $j$th volume groups.

- **Dummy variables** $\bar{q}_j, j = 1, ..10$ quantify the average trade size for a benchmark stock based on data for $j$th volume group. If assumptions of the model are reasonable, then all dummy variables should be constant across volume groups.
Average Order Sizes Across Volume Groups

Figure plots average order size $\bar{q}_j$ across 10 volume groups. Group 1 contains stocks with the lowest volume. Group 10 contains stocks with the highest volume. The volume thresholds are 30th, 50th, 60th, 70th, 75th, 80th, 85th, 90th, and 95th percentiles for NYSE stocks.
Tests for Orders Size - Summary

**Predictions:** If the data match assumptions well, then all dummy variables $\bar{q}_j, j = 1, .10$ should be constant across volume groups.

**Results:** The data match the assumptions of Model of Trading Game Invariance much better than the two alternative models.

**Discussion:**
- Pattern of dummy variables of Model of Trading Game Invariance is reasonably constant.
- But note that in Model of Trading Game Invariance, trade size for largest 5% of stocks is statistically larger than predicted by the model, due to low standard errors.
- Alternative models fail miserably to explain the data on trade sizes.
Distribution of Order Sizes

Trading games invariance predicts that distributions of order sizes $X$, adjusted for differences in trading activity $W$, are the same across different stocks:

$$\ln \left( \frac{|\tilde{Q}|}{V} \cdot \frac{1}{W^{-2/3}} \right).$$

We compare these distributions across 10 volume and 5 volatility groups.
Distributions of Order Sizes

Trading game invariance works well for entire distributions of order sizes. These distributions are approximately log-normal.
Tests for Orders Size - Conclusion

Data on the sizes of portfolio transition orders strongly support assumptions made in Model of Trading Game Invariance. The data soundly reject assumptions made in alternative models.

Intuition: when trading activity increases, both frequency and size of trades increase; neither remains constant.
Portfolio Transitions and Trading Costs

We use data on the implementation shortfall of portfolio transition trades to test predictions of the three proposed models concerning how transaction costs, both market impact and bid-ask spread, vary with trading activity.
Portfolio Transitions and Trading Costs

“Implementation shortfall” is the difference between actual trading prices (average execution prices) and hypothetical prices resulting from “paper trading” (price at previous close).

There are several problems usually associated with using implementation shortfall to estimate transactions costs. Portfolio transition orders avoid most of these problems.
Problem I with Implementation Shortfall

Implementation shortfall is a biased estimate of transaction costs when it is based on price changes and executed quantities, because these quantities themselves are often correlated with price changes in a manner which biases transactions costs estimates.

**Example A:** Orders are often canceled when price runs away. Since these non-executed, high-cost orders are left out of the sample, we would underestimate transaction costs.

**Example B:** When a trader places an order to buy stock, he has in mind placing another order to buy more stock a short time later.

For **portfolio transitions**, this problem does not occur: Orders are not canceled. The timing of transitions is somewhat exogenous.
Problems II with Implementation Shortfall

The second problem is statistical power.

Example: Suppose that 1% ADV has a transactions cost of 20 bps, but the stock has a volatility of 200 bps. Order adds only 1% to the variance of returns. A properly specified regression will have an R squared of 1% only!

For portfolio transitions, this problem does not occur: Large and numerous orders improve statistical precision.
Tests For Market Impact and Spread - Design

All three models are nested into one specification that relates trading activity $W$ and implementation shortfall $C$ for a transition order for $X$ shares:

$$C_i \cdot \left[ \frac{0.02}{\sigma} \right] = \frac{1}{2} \bar{\lambda} \cdot \left[ \frac{W_i}{W_*} \right]^{\alpha_0} \cdot \frac{X_i}{(0.01)V_i} + \frac{1}{2} \bar{k} \cdot \left[ \frac{W_i}{W_*} \right]^{\alpha_1} \cdot \frac{(X_{omt,i} + X_{ec,i})}{X_i} + \tilde{\epsilon}$$

The variables are scaled so that parameters $\bar{\lambda}$ and $\bar{k}$ measure in basis point the market impact (for 1% of daily volume $V$) and spread for a benchmark stock with volatility 2% per day, price $40 per share, and daily volume of 1 million shares.

- Spread is assumed to be paid only on shares executed externally in open markets and external crossing networks, not on internal crosses.
- Implementation shortfall is adjusted for differences in volatility.
Tests For Market Impact and Spread - Design

The three models make different predictions about parameters $a_0$ and $a_1$.

- **Model of Market Microstructure Invariants:**
  $\alpha_0 = 1/3, \alpha_1 = -1/3$.

- **Model of Invariant Bet Frequency:** $\alpha_0 = 0, \alpha_1 = 0$.

- **Model of Invariant Bet Size:** $\alpha_0 = 1/2, \alpha_1 = -1/2$.

We estimate $a_0$ and $a_1$ to test which of three models make the most reasonable predictions.
## Tests For Market Impact and Spread - Results

<table>
<thead>
<tr>
<th></th>
<th>All</th>
<th>NYSE</th>
<th>NASDAQ</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Buy</td>
<td>Sell</td>
</tr>
<tr>
<td>( \frac{1}{2} \bar{\lambda} )</td>
<td>2.85***</td>
<td>2.50*** 2.33***</td>
<td>4.2*** 2.99***</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.245)</td>
<td>(0.515) (0.365)</td>
</tr>
<tr>
<td>( \alpha_0 )</td>
<td>0.33***</td>
<td>0.18*** 0.33***</td>
<td>0.33*** 0.35***</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.024)</td>
<td>(0.045) (0.054)</td>
</tr>
<tr>
<td>( \frac{1}{2} \bar{k} )</td>
<td>6.31***</td>
<td>14.99*** 2.82*</td>
<td>8.38* 3.94**</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1.131)</td>
<td>(2.529) (1.394)</td>
</tr>
<tr>
<td>( \alpha_1 )</td>
<td>-0.39***</td>
<td>-0.19*** -0.46***</td>
<td>-0.36*** -0.45***</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.025)</td>
<td>(0.045) (0.061)</td>
</tr>
</tbody>
</table>

- Model of Market Microstructure Invariants: \( \alpha_0 = 1/3, \alpha_1 = -1/3 \).
- Model of Invariant Bet Frequency: \( \alpha_0 = 0, \alpha_1 = 0 \).
- Model of Invariant Bet Size: \( \alpha_0 = 1/2, \alpha_1 = -1/2 \).

*** is 1%-significance, ** is 5%-significance, * is 10%-significance.
## Tests For Market Impact and Spread - F-Tests

<table>
<thead>
<tr>
<th></th>
<th>NYSE</th>
<th>NASDAQ</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>All</td>
<td>Buy</td>
</tr>
<tr>
<td>Model of Market Microstructure Invariants: $\alpha_0 = 1/3, \alpha_1 = -1/3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>F-test</td>
<td>2.60</td>
<td>8.57</td>
</tr>
<tr>
<td>p-val</td>
<td>0.0742</td>
<td>0.0002</td>
</tr>
</tbody>
</table>

Model of Invariant Bet Frequency: $\alpha_0 = 0, \alpha_1 = 0$

|                      |                           |     |      |     |      |
| F-test               | 176.14                    | 14.77 | 47.03 | 33.11 | 71.06 |
| p-val                | 0.0000                    | 0.0000 | 0.0000 | 0.0000 | 0.0000 |

Model of Invariant Bet Size: $\alpha_0 = 1/2, \alpha_1 = -1/2$

|                      |                           |     |      |     |      |
| F-test               | 30.34                     | 39.81 | 5.23 | 7.21 | 5.92 |
| p-val                | 0.0000                    | 0.0000 | 0.0054 | 0.0007 | 0.0027 |
Tests for Impact and Spread - Summary

Model of Market Microstructure Invariants predicts: The coefficient for market impact is $\alpha_0 = 1/3$. The coefficient for bid-ask spread is $\alpha_1 = -1/3$.

Results: Coefficient $\alpha_0$ is estimated to be 0.33, matching prediction of Market Microstructure Invariants exactly. Coefficient $\alpha_1$ is estimated to be $-0.39$, matching prediction of the model reasonably closely.

Discussion:

- Model of Market Microstructure Invariants is statistically rejected due to small standard errors and imperfect match for spread.
- Alternative models are soundly rejected.
- For benchmark stock, half-spread is 7.90 basis points and half market impact is 2.89 basis points (restricting $\alpha_0$ to be 1/3 and $\alpha_1$ to be -1/3).
Transactions Costs Across Volume Groups

Do the data match models’ assumptions across 10 volume groups?

\[ C_i \cdot \left[ \frac{0.02}{\sigma} \right] = \left( \sum_{j=1}^{10} \mathbb{I}_{j,i} \cdot \frac{1}{2} \lambda_j \right) \left[ \frac{W_i}{W_*} \right]^{\alpha_0} \frac{X_i}{(0.01)V_i} + \left( \sum_{j=1}^{10} \mathbb{I}_{j,i} \cdot \frac{1}{2} k_j \right) \left[ \frac{W_i}{W_*} \right]^{\alpha_1} \frac{(X_{omt,i} + X_{ec,i})}{X_i} + \tilde{\epsilon} \]

- **Parameter** \( \alpha_0 \) and \( \alpha_1 \) are restricted to values predicted by each model
  \( (\alpha_0 = 1/3, \alpha_0 = -1/3; \alpha_0 = 0, \alpha_0 = 0; \text{ or } \alpha_0 = 1/2, \alpha_0 = -1/2) \).

- **Indicator variable** \( \mathbb{I}_{j,i} \) is one if \( i \)th order is in the \( j \)th volume groups.

- **Dummy variables** \( \bar{\lambda}_j \) and \( \bar{k}_j, j = 1, \ldots 10 \) quantify the market impact and spread for \( j \)th volume group. If assumptions of the model are reasonable, then all dummy variables should be constant across volume groups.
Figure plots half market impact $\frac{1}{2} \bar{\lambda}_j$ and half effective spread $\frac{1}{2} \bar{k}_j$ across 10 volume groups. Group 1 contains stocks with the lowest volume. Group 10 contains stocks with the highest volume. The volume thresholds are 30th, 50th, 60th, 70th, 75th, 80th, 85th, 90th, and 95th percentiles for NYSE stocks.
Tests for Impact and Spread - Summary

Predictions: If the data match predictions well, then all dummy variable $\bar{\lambda}_j$ and $\bar{k}_j, j = 1,..10$ should be constant across volume groups.

Results: Pattern is more stable for our model of Market Microstructure Invariants than for other two models.

Discussion:

- High precision for small stock anchors models parameters.
- For model of Market Microstructure Invariants, most active stocks have less impact and higher spreads than predicted, due to basket trades?
- Model of Invariant Bet Frequency gives more weight to orders in small stocks (since these orders are large relative to volume) and incorrectly extrapolates the estimates for small stocks to large ones. This model does reasonably when small stocks are excluded from the sample.
Conclusions

Our tests provide strong support for the model of Market Microstructure Invariants which implies, for example, that a one percent increase in trading activity \( W = V \cdot P \cdot \sigma \) is associated with ... 

- an increase of \( \frac{1}{3} \) of one percent in average order size,
- an increase of \( \frac{2}{3} \) of one percent in its arrival frequency,

and leads to...

- an increase of \( \frac{1}{3} \) of one percent in market impact,
- a decrease of \( \frac{1}{3} \) of one percent in bid-ask spread.
Practical Implications

For a benchmark stock, half market impact $\frac{1}{2}\lambda^*$ is 2.89 basis points and half-spread $\frac{1}{2}k^*$ is 7.90 basis points.

The Model of Market Microstructure Invariants extrapolates these estimates and allows us to calculate expected trading costs for any order of $X$ shares for any security using a simple formula:

$$C(X) = \frac{1}{2}\lambda^* \left( \frac{W}{(40)(10^6)(0.02)} \right)^{1/3} \frac{\sigma}{0.02} \frac{X}{(0.01)V} + \frac{1}{2}k^* \left( \frac{W}{(40)(10^6)(0.02)} \right)^{-1/3} \frac{\sigma}{0.02},$$

where trading activity $W = \sigma \cdot P \cdot V$

- $\sigma$ is the expected daily volatility,
- $V$ is the expected daily trading volume in shares,
- $P$ is the price.
1987 Stock Market Crash

Facts about 1987 stock market crash:

- **Trading volume** on October 19 was **$40 billion** ($20 billion futures plus $20 billion stock). Typical volume was lower (say $20 billion) but inflation makes 1987 dollar worth more than 2001-2005 dollar.

- **Volatility** during crash was extremely high, so **2%** expected volatility per day might be reasonable.

- From Wednesday to Tuesday, **portfolio insurers** sold **$14 Billion** ($10 billion futures plus $4 billion stock).

- From Wednesday to Tuesday, **S&P 500 futures** declined from 312 to 185, a decline of **41%** (including bad basis). **Dow** declined from 2500 to 1700, a decline of **32%**.
1987 Stock Market Crash

Our market impact formula implies decline of

$$2 \cdot 2.89 \cdot \left( \frac{40 \cdot 10^9}{40 \cdot 10^6} \right)^{1/3} \cdot \frac{0.02}{0.02} \cdot \frac{14/40}{0.01} = 20.23\%$$

Our model suggests portfolio insurance selling had market impact of about 20%. Keep in mind that assumptions are approximations, so result is an approximation as well.
Fraud at Société Générale, January 2008

Facts about a fraud:

- From Jan 21 to Jan 23, a fraudulent position of Jérôme Kerviel had to be liquidated: €30 billion in STOXX50 futures, €18 billion in DAX futures, and €2 billion in FTSE futures.

- Trading volume was €61 billion in STOXX50 futures, €38 billion in DAX futures, and €8 billion in FTSE futures.

- Volatility was about 1% per day.

- Bank has reported exceptional losses of €6.4 billion, which were attributed to “adverse market movements” between Jan 21 and Jan 23.
Fraud at Société Générale, January 2008

Our market impact formula implies a total liquidation costs of about €3.60 billion.

If we take into account losses on Kerviel’s position during the market’s decline between 31 Dec 2007 and 18 Jan 2008 (estimated to range between €2 billion and €4 billion), we conclude that our estimates are consistent with reported losses of €6.4 billion.
The “Flash Crash” of May 6, 2010

News media report that a large trader sold 75,000 S&P 500 E-mini contracts. Prices fell approximately 300 bp during first part of day, then suddenly fell about 500 bp over 10 minutes, then rose about 500 bp over next 10 minutes.

- One contracts represents ownership of about $55,000 with S&P level of 1100.
- Typical contract volume was about 2 million contracts per day, or $110 billion (but much higher on May 6).
- Volatility was high due to European debt crisis; rough estimate is $\sigma = 0.02$.

Our market impact formula implies decline of

$$2 \cdot 2.89 \cdot \left( \frac{110 \cdot 10^9}{40 \cdot 10^6} \right)^{1/3} \cdot \frac{0.02}{0.02} \cdot \frac{75 \cdot 10^3}{2 \cdot 10^6 \cdot 0.01} = 303.67 \text{bp}$$

Flash crash research in progress by Kirilenko, Kyle, Tuzun, and Samadi.
More Practical Implications

- **Trading Rate:** If it is reasonable to restrict trading of the benchmark stock to say 1% of average daily volume, then a smaller percentage would be appropriate for more liquid stocks and a larger percentage would be appropriate for less liquid stocks.

- **Components of Trading Costs:** For orders of a given percentage of average daily volume, say 1%, bid-ask spread is a relatively larger component of transactions costs for less active stocks, and market impact is a relatively larger component of costs for more active stocks.

- **Comparison of Execution Quality:** When comparing execution quality across brokers specializing in stocks of different levels of trading activity, performance metrics should take account of nonlinearities documented in our paper.
Evidence From TAQ Dataset Before 2001

Trading game invariance seems to work in TAQ before 2001, subject to market frictions (Kyle, Obizhaeva and Tuzun (2010)).
Trading game invariance is **hard to test** in TAQ after **2001**.
News and Trading Game Invariance

Data on the number of Reuters news items $N$ is consistent with trading game invariance (research in progress by Kyle, Obizhaeva, Ranjan, and Tuzun (2010)).
More Philosophical Implications

Trades and prices are not completely random. There are similar structures, i.e. “trading games”, in the trading data. Trading games are invariant across stocks and across time, except they are played at different pace.