

Enforcement, Incomplete Contracts, and Firm Dynamics

Cristina Arellano^{*}, Yan Bai^Y and Jing Zhang^{**}

University of Minnesota and Federal Reserve Bank of Minneapolis, Arizona State
University, University of Michigan

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Abstract

This paper studies the relation between firm size, leverage, exit and growth in an environment of incomplete markets and lack of enforcement. In the data large firms have lower growth and exit rates yet higher leverage. The idea of the paper is that high productivity firms will be larger and have a greater degree of enforcement because their outside option relative to the value of their productive project is large. Thus big firms can sustain larger levels of debt and have larger leverage ratios without having incentives to default. However with incomplete markets a highly productive firm that receives a sequence of bad shocks will shrink over time as its debt increases and its value decreases. Thus highly leveraged firms are also more likely to exit if their high debt levels are due to a history of bad shocks. We quantify both mechanisms and calibrate our model to the firm data in Ecuador and find that our mechanism provides a unified rationale for the relation between leverage, exit and size.

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^{*} arellano@econ.umn.edu

^Y yan.bai@asu.edu

^{**} jzhang@umich.edu

1. Introduction

Firm size is correlated with growth, exit and leverage. Large firms tend to have lower exit rates, lower growth rates and larger debt to assets ratios. The relation between size, growth and exit has been documented extensively for U.S. firms.¹ Rajan and Zingales (1995) document the leverage-size relation for publicly traded firms in the OECD countries. They find that in most of OECD countries large firms have larger debt to asset ratios. We document the relation between size, growth, exit and leverage for a new dataset of firms in an emerging market Ecuador. We document similar patterns as in the U.S. In Ecuador small firms have larger exit and growth rates, and smaller leverage ratios. However conditional on size firms that exit, have more debt relative to assets.

This paper builds a model of heterogeneous firms to study the link between enforcement in financial contracts and firms' dynamics. In particular we study how enforcement problems and incomplete markets can provide a rational for the facts regarding leverage, exit, growth and size. The idea of the paper is that high productivity firms will be larger and have a greater degree of enforcement because their outside option relative to the value of their productive project is large. Thus big firms can sustain larger levels of debt and have larger leverage ratios without having incentives to default. However with incomplete markets a highly productive firm that receives a sequence of bad shocks will shrink over time as its debt increases and its value decreases. Thus highly leveraged firms are also more likely to exit if their high debt levels are due to a history of bad shocks. We quantify both mechanisms and calibrate our model to the firm data on Ecuador and find that our mechanism provides a unified rational for the relation between leverage, exit and size.

The framework is a dynamic model of heterogeneous firms similar to Albuquerque and Hopenhayn (2004) but under incomplete markets. Firms in the model borrow from foreign investors to finance the working capital and set up costs needed for production. Firms' productivity consists of two components: a permanent component and an i.i.d. component. We assume that firms sign contracts with investors to finance working capital before their i.i.d. shock is known and that these contracts cannot be contingent on the shock realization. After

¹ See for example Rossi-Hansberg and Wright (2005).

observing their shock firms can choose to repay the capital borrowed and the debt due and remain in operation for the next period, or default, exit, and get a new draw for a project with some probability. Incentives to repay debt depend crucially on the value of keeping the firm in operation at every period relative to the value of redrawing a new project.

Among firms with only temporary productivity differences the firm's value depends inversely on the amount of debt that it owes to creditors. We show that firms with low values and high debt are inefficiently small because they under-invest relative to an unconstrained first best level. Enforcement problems are severe for low value firms, limiting the amount of risk-free funds. If firms instead borrow risky and default in some states, the high interest rates charged on loans also distort downward the optimal investment and the size of firms. Thus low value firms uniformly underinvest and produce at inefficient scales. Incomplete contracts introduce rich dynamics in the debt firms owe and the value of firms. In our model firms with a history of bad shocks accumulate debt, while reducing its value because their low output is not enough to cover interest payments on outstanding debt. In fact with enough bad shock realizations the firm with high debt chooses to default and exit. Thus our model generates 'positive persistence' for low productivity realizations in firms' output because as debt grows over time the inefficiency in scale worsens and output further contracts. Incompleteness of contracts and enforcement problems are key for generating the result that firms decrease their value and are likely to exit after a sequence of bad shocks.

Firms with permanent productivity differences have a distinct long run degree of enforcement because of the larger value of more productive projects relative to a constant default value of re-drawing a project. The surplus from the relation with lenders is larger for firms with high productivity and thus the maximum sustainable debt is larger. We show that without uncertainty and permanent productivity differences, our model delivers a monotonic ranking between borrowing limits and leverage ratios with size. More productive firms have larger assets and larger debt than less productive firms. But more productive firms can sustain disproportional larger debt to assets ratios because the value of the project is not only larger due to the larger assets but because it is much larger than the outside default value of re-drawing a new project.

We then calibrate the model to match certain features of the firm size distribution in Ecuador. The important parameters are the permanent versus transitory components of the productivity for firms because both give distinct usage for debt within the context of our model. We find that enforcement friction and incomplete markets can deliver the features of the data that larger firms have larger leverage ratios, yet conditional on size firms that exit have higher leverage. The model also delivers a higher exit rate and growth rate for small firms as in the data.

The paper is related to the literature that studies the implications of financial frictions for the dynamics and firm size distribution. Cooley and Quadrini (2001) study how financial frictions can rationalize the relation between exit and growth with size. Our model shares many of the features of their paper, however we are concentrating on how enforcement frictions can help explain the relation between leverage and size. Moreover we focus on the distinct implications for debt financing in the presence of permanent and temporary productivity differences across firms and we abstract from equity issuances. Albuquerque and Hopenhayn (2004) focus on the effects of enforcement problems and solve for the optimal state contingent contract. Our environment is different from them in that we consider an incomplete set of assets. Incomplete markets allows for firms in the model with a history of bad shock to decrease the effective degree of enforcement through time by increasing their debt holdings.

Clementi and Hopenhayn (2005) and Quadrini (2004) are also papers that study financial imperfections and firm dynamics. These papers study financial constraints that arise due to informational asymmetries between the lender and the entrepreneur and show that moral hazard considerations can also rationalize borrowing constraints which make investment sensitive to cash flows. Also information asymmetries can also provide a rational for the relation between growth and size.

2. Empirical Facts

[TO BE COMPLETED]

Data for balance sheets of 15,000+ firms in Ecuador for years 1996, 1997, 1998 and 1999 from Local Regulatory Agency

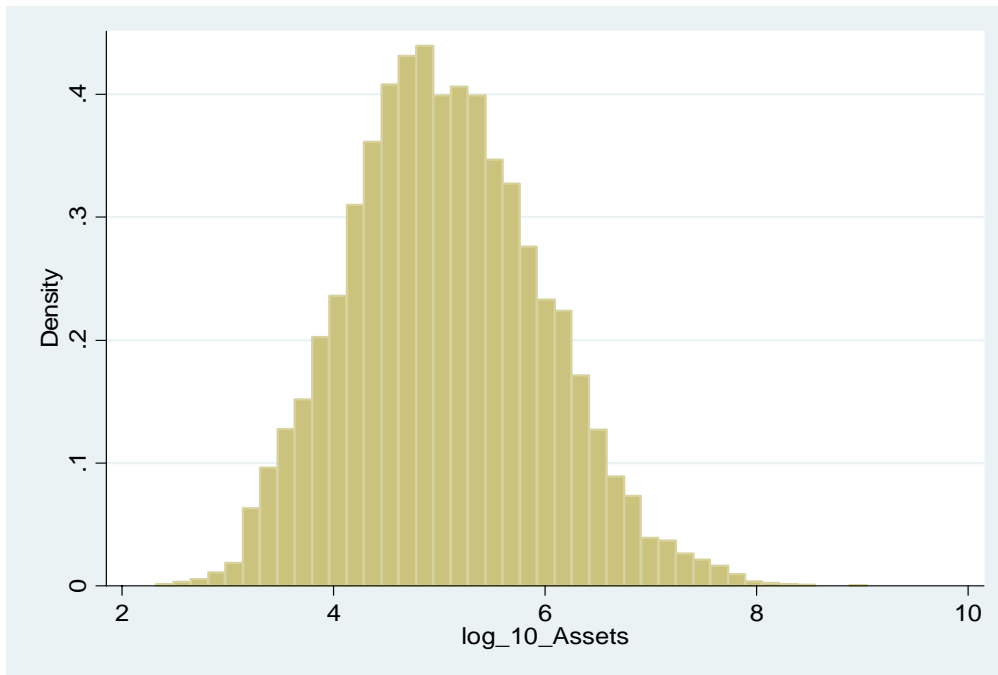


Figure 1. Asset Distribution in 1996

Table 1. Empirical Facts on Firm Dynamics and Size

	Median Asset	Growth	Exit	Leverage	Leverage of Exit
Overall	\$93,300	0.51	0.07	0.61	0.70
0.0-0.2 percentile	\$6,647	1.27	0.13	0.50	0.57
0.2-0.4 percentile	\$30,895	0.56	0.08	0.59	0.66
0.4-0.6 percentile	\$93,286	0.40	0.06	0.63	0.76
0.6-0.8 percentile	\$311,556	0.35	0.05	0.66	0.76
0.8-1.0 percentile	\$1,848,652	0.34	0.06	0.68	0.74

3. Model Economy

The economy consists of a small country populated by a continuum of infinitively lived entrepreneurs who have access to risky projects and finance those projects by borrowing from foreign lenders. Financial contracts are not enforceable and entrepreneurs can default on the debt they owe to creditors. Contracts in this economy are also incomplete and thus default can occur along the equilibrium.

3.1 Firms

Entrepreneurs in the economy have access to a mass of N project opportunities to produce consumption goods. We define each project opportunity as a firm. Every period a fraction ξ of the projects are available to new entrepreneurs. Entrepreneurs are risk neutral, infinitively lived and decide on entry, exit and production and financing plans to maximize the lifetime value of dividends from the project. Entrepreneurs discount time at rate $\beta < 1$ and face an exogenous probability of dying $\delta < 1$ each period. New entrepreneurs have no initial wealth and their outside option is equal to 0. Thus, it is always worth it for an entrepreneur to enter and operate the firm if he draws a project that gives him positive expected present value of dividends. Each entrepreneur owns at most one firm.

Firms are funded by foreign lenders who offer contracts to cover two distinctive needs: initial set up costs and working capital requirements. Each new firm needs to pay an initial set-up cost I to start its operation. Since new firms have no initial wealth, they borrow from foreign long-lenders this initial set up cost. The set up cost is the initial loan for every firm $B = I$. This initial loan is associated with a starting long-debt position for the firm $B_R = I_R$ that is due the next period. Every subsequent period the firm can choose a new loan and long-debt position (B', B'_R) from the set of long term contracts offered. By choosing a different long-term contract corresponding to a smaller long debt position, $B'_R < B_R$, the firm effectively repays off part of its initial set up cost.

Every period each firm that is in operation produces output y with a stochastic production technology that transforms capital goods into consumption goods with a decreasing returns technology. For a given level of capital input K invested at the beginning of the period, the firm produces output y at the end of the period such that

$$y = z\varepsilon_z K^\alpha, \quad (1)$$

where $0 < \alpha < 1$. The productivity shock has two firm specific components. One is the permanent productivity level z , which is drawn from distribution $F(z)$. The other is the temporary productivity level ε_z , which is i.i.d. and drawn from distribution $G_z(\varepsilon_z)$. Capital depreciates completely every period.

The timing of decisions within the period is as follows. At the beginning of the period, each entrepreneur who owns a project with long-debt B_R , decides how much capital K to invest in the project depending on his permanent productivity level and his expectation of temporary productivity shocks. Firms borrow the working capital desired for production from foreign short-lenders. At the end of the period the temporary productivity shock ε_z is realized and the firm decides whether to repay or default on all debt due. If the firm repays the working capital and the long debt, then it chooses a long-debt contract from the ones offered and continues operations for the next period. If the firm defaults on all the debt due, both short and long term, then it gets part of their output, and exits. Every firm that exits frees up an additional project opportunity for a new entrepreneur the following period.

3.2 Contracts

The set of contracts between the firm and lenders are of two types: short-term contracts and long-term contracts. Short-term contracts are for capital requirements. These are traded in the beginning of the period before the temporary productivity shock is known and consist of two numbers (K, K_R) . K denotes the payment in the beginning of the period from the lender to the firm which is used as capital input for producing output. K_R is the payment the firm promises the lender at the end of the period conditional on not defaulting. The set of short-

term contracts are a function of the firms' long-debt position in the beginning of the period B_R and its permanent productivity level z .

Long-term contracts are traded at the end of the period after the productivity shock is known and consist also of two numbers (B', B'_R) . B' is the immediate payment from the lender to the firm that is used for interest payments of the original set up costs and to increase period dividends. B'_R is payment the firm promises the lender next period conditional on not defaulting. B'_R is the new long debt position of the firm next period. The set of long-term contracts are independent of the firm's long debt in the beginning of the period B_R , but depends on the firm's permanent productivity level z .² Both contracts are exogenously incomplete in that they are not a function of the output realization when the payments are due.

3.3 Incentives, Dividends and Production

The dividend each entrepreneur receives at the end of the period depends on whether he decides to repay his debts and remain in operation, or default and exit. After observing the temporary productivity shock ε_z at the end of the period, if repayment is chosen, then the entrepreneur of each firm receives as dividend

$$d = z\varepsilon_z K^\alpha - K_R - B_R + B' . \quad (2)$$

The dividend conditional on repayment equals the output produced with capital K in the period minus payment of the short-term working capital debt K_R and long debt due B_R plus new long debt issuances B' . Also we define as 'profit' the period return from the project excluding long debt operations: $z\varepsilon_z K^\alpha - K_R$.

If the entrepreneur chooses to default on his debt then the firm exits, the entrepreneur loses the project. After default the entrepreneur can get some additional benefit V^d . This additional benefit represents any opportunities and payoffs the entrepreneur might get after clos-

² We have written the long-term contract as a one period debt contract for simplicity. It can be easily shown that these contracts can be written as true long-debt contracts where firms change their stock of long debt every period.

ing the firm. Here we model the default option as getting a draw for a new project from $F(z)$ with probability θ . The value of default for the entrepreneur is then denoted by

$$V^d = \theta \int_z W(z, I_R) dF(z),$$

where $W(z, I_R)$ denotes the welfare of the starting firm with the permanent productivity z , and I_R is the corresponding long contract repayment for loan size I . The parameter θ controls the degree of enforceability of financial contracts in the country. If θ is very low default is very unattractive for entrepreneurs and we interpret this as the case with higher enforceability of financial contracts.

Given that the entrepreneur's only financial relations are with the short-term lender and the long-term lender we also impose a limited liability condition such that

$$d \geq 0. \quad (3)$$

The decision to default or repay debts depends on the current dividend the entrepreneur receives and on the entire future stream of dividends the project will generate. The firm will experience two types of default events: voluntary and involuntary. We first discuss the voluntary default decision.

The voluntary default decision for a particular firm that has a permanent productivity z , long debt B_R and short contract (K, K_R) after observing its temporary productivity shock ε_Z is given by

$$W^o(B_R, K, K_R, z, \varepsilon_Z) = \max \left\{ W^c(B_R, K, K_R, z, \varepsilon_Z), V^d \right\}, \quad (4)$$

where $W^o(B_R, K, K_R, z, \varepsilon_Z)$ denotes the present value of this firm and $W^c(B_R, K, K_R, z, \varepsilon_Z)$ the present value of such firm conditional on repaying debt today, but that tomorrow has the option to default.

If the entrepreneur decides to repay his debt, then he chooses a long-term contract (B', B'_R) to maximize the value of staying in the contract

$$W^c(B_R, K, K_R, z, \varepsilon_Z) = \max_{B', B'_R} \left(z \varepsilon_Z K^\alpha - K_R - B_R + B' + \beta(1 - \delta)W(B'_R, z) \right). \quad (5)$$

$W(B'_R, z)$ is the present value of a firm with debt B'_R and a permanent productivity z in the beginning of the following period before the temporary productivity shocks are realized. By paying the short-term debt and choosing the new long-term contract the entrepreneur can remain in operation for one more period. The entrepreneur understands that the decision to repay debts is a period-by-period decision and that tomorrow he will again have the option to default.

The second type of default firms will experience is involuntary. Under this class of default event, the firm cannot satisfy its non-negative dividend condition with the set of long contracts available and it is forced to default although the lifetime value of remaining in the contract is larger. Formally, a firm that has long debt B_R , short contract (K, K_R) and productivity $z\varepsilon_Z$ is forced to default if

$$d = z\varepsilon_Z K^\alpha - K_R - B_R + B' < 0 \text{ for all } (B', B'_R).$$

The default policy of each firm can be summarized by repayment sets and default sets. For a given initial level of debt B_R and short contract is (K, K_R) , denote the involuntary default set $D_{inv}(B_R, K, K_R, z)$ as the ε_Z set for which the firm cannot have positive dividends with any of the long contracts available

$$D_{inv}(B_R, K, K_R, z) = \{\varepsilon_Z : z\varepsilon_Z K^\alpha - K_R - B_R + B' < 0 \forall (B', B'_R)\}. \quad (6)$$

Voluntary default sets $D_{vol}(B_R, K, K_R, z)$ are the set of ε_Z for which the value under the contract is less than the value under default

$$D_{vol}(B_R, K, K_R, z) = \{\varepsilon_Z \in E_Z / D_{inv}(B_R, K, K_R, z) : W^c(B_R, K, K_R, z, \varepsilon_Z) < V^d\}. \quad (7)$$

The default set is then the union of the voluntary default set and the involuntary default set

$$D(B_R, K, K_R, z) = D_{vol}(B_R, K, K_R, z) \cup D_{inv}(B_R, K, K_R, z). \quad (8)$$

The repayment set is defined as the complement of the default set:

$$A(B_R, K, K_R, z) = [D(B_R, K, K_R, z)]^c.$$

Besides debt and default choices, the entrepreneur makes investment decisions every period. In the beginning of the period before the productivity shock is observed the entrepreneur decides on capital and production plans. Given an initial level of debt B_R the entrepreneur chooses the short-term contract (K, K_R) to finance investment such that it maximizes his value

$$W(B_R, z) = \max_{K, K_R} \int W^o(B_R, K, K_R, z, \varepsilon_Z) dG_Z(\varepsilon_Z). \quad (9)$$

The decision of how much capital to invest depends on the permanent productivity level, the expectations of average temporary productivity and on the set of contracts available. For example, if temporary productivity is expected to be high, the optimal size of the project is larger and the entrepreneur has an incentive to invest a large level of capital. In fact in a world without enforcement frictions, the entrepreneur will choose the size of the project proportionally to temporary productivity expectations. However with enforcement frictions the optimal size of the project can be distorted by the set of short-term and long-term contracts available. In particular if the firm already starts the period with a large level of long term debt then default would be more likely and thus the set of short term and long term contracts will very limited. The entrepreneur might then not be able to run his project with an optimal size and the project will be inefficiently small. We show in the next session that the link between the level of long debt and the scale of project imply that firms with high debt under-invest because enforcement problems are very stringent.

3.4 Lenders

Lenders in the model are assumed to be competitive and discount time at the rate of the international risk free interest rate r . They behave passively and are willing to finance the firm's initial set up costs and working capital needs as long as they are compensated for the expected loss in case of default. In particular a short-term lender offers contracts (K, K_R) to a firm with long debt B_R such that

$$K = \frac{K_R}{(1+r)} \left(1 - \int_{D(B_R, K, K_R, z)} dG_Z(\varepsilon_Z) \right). \quad (10)$$

With every short term contract (K, K_R) the lender breaks even in expected value.

A long-term lender offers contracts (B', B'_R) such that with every contract the lender receives in expectation the risk free interest rate

$$B' = \frac{B'_R}{1+r} \left(1 - \int_{D(B'_R, K'(B'_R, z), K'_R(B'_R, z))} dG_Z(\varepsilon_Z) \right). \quad (11)$$

Every long-term contract offered (B', B'_R) forecasts the firm's choice of the short-term contract next period $(K'(B'_R, z), K'_R(B'_R, z))$. This is because the firm's decision to default or repay is carried out after the capital stock is in place and the productivity is realized.

When a new firm starts its operation, its initial long-term contract (I, I_R) is such that the firm receives as first payment the initial set-up costs needed I with

$$I = \frac{I_R(1-\delta)}{1+r} \left(1 - \int_{D(I_R, K(I_R, z), K_R(I_R, z))} dG_Z(\varepsilon_Z) \right). \quad (12)$$

If initial set up costs are so high then there might not exist a long-term contract that can finance the entry of the firm because probabilities of default are too high.

3.5 Equilibrium

We now define the equilibrium:

Definition.

The recursive equilibrium for this economy is defined as: (i) the policy functions for short-term contracts $(K(B_R, z), K_R(B_R, z))$, long-term contracts $(B'(B_R, K, K_R, z, \varepsilon_Z), B'_R(B_R, K, K_R, z, \varepsilon_Z))$, dividends $d(B_R, K, K_R, z, \varepsilon_Z)$, repayment sets $A(B_R, K, K_R, z)$ and default sets $D(B_R, K, K_R, z)$ for each firm, (ii) a menu of short-term contracts

$(K(B_R, z), K_R(B_R, z))$ and long-term contracts $(B'(z), B'_R(z))$ offered to each firm given its productivity, (iii) a distribution of firms over permanent productivity level, temporary productivity shocks and long debt holdings $\Upsilon(B_R, z, \varepsilon_z)$ and (iv) a mass of new entrants $\xi(\Upsilon(B_R, z, \varepsilon_z))$, such that:

1. Taking as given the menu of short-term contracts $(K(B_R, z), K_R(B_R, z))$, and long-term contracts $(B'(z), B'_R(z))$ offered, the policy functions $(K(B_R, z), K_R(B_R, z))$, $(B'(B_R, K, K_R, z, \varepsilon_z), B'_R(B_R, K, K_R, z, \varepsilon_z))$, $d(B_R, K, K_R, z, \varepsilon_z)$, repayment sets $A(B_R, K, K_R, z)$ and default sets $D(B_R, K, K_R, z)$ satisfy the firm's optimization problem.
2. Short-term contracts and long-term contracts available to each firm reflect the firm's default probabilities such that lenders break even in expected value.
3. The distribution of firms $\Upsilon(B_R, z, \varepsilon_z)$ is consistent with individual decisions and shocks.
4. The mass of new entrants $\xi(\Upsilon(B_R, z, \varepsilon_z))$ is equal to the measure of all the firms that default and die in the limiting distribution $\Upsilon(B_R, z, \varepsilon_z)$.

3.6 First Best Benchmark

We will compare our model to the benchmark where contracts are perfectly enforceable but incomplete. We denote this benchmark model as the first best. The capital stock of each firm in the first best is such that the expected marginal product of capital equals the interest rate. Firms have different sizes because of different permanent productivity levels. The firm pays each period the perpetuity value of the set-up cost and receives as dividend the residual. In the first best the capital stock is determined by

$$zE(\varepsilon_z)\alpha K_{fb}(z)^{\alpha-1} = (1+r). \quad (13)$$

Though we have different sizes of firms, all the firm are producing at the efficient scales given their permanent productivity level.

The first best short contract is given by

$$\left\{ K_{fb}(z), K_{Rfb}(z) \right\} = \left\{ \left(\frac{zE(\varepsilon_Z)\alpha}{1+r} \right)^{\frac{1}{1-\alpha}}, (1+r) \left(\frac{zE(\varepsilon_Z)\alpha}{1+r} \right)^{\frac{1}{1-\alpha}} \right\}.$$

Given the model setup, we will next illustrate how the enforcement friction interacts with firms' productivity shocks under incomplete contracts. Especially we will focus on the stylized facts documented in Section 2. In the next two sections, we will use some numerical exercises with the temporary shocks alone or the permanent productivity differences alone to study model mechanisms firm dynamics. Section 6 will study the quantitative implications of the model by calibrating model parameters and conducting simulation.

4. Firms Dynamics with Temporary Shocks Only

This section studies the implications from lack of enforcement in contracts and incomplete markets in terms of firm size and output dynamics and as well as the size-growth, size-leverage and size-exit relations with only the temporary productivity shocks. We find are that in our model firms with high levels of long debt are inefficiently small, and this inefficiency is exacerbated over time if firms receive a sequence of bad productivity realizations. Our model also delivers a limiting distribution of firms with inefficient scales relative to the first best benchmark due to the debt and size relationship. Moreover the model reproduces some statistics of the firm dynamics in the data, such as the size-growth and size-exit relationship, though the size-leverage relationship is the opposite of the data.

4.1 Underinvestment

We first characterize default and investment decisions for firms. The temporary productivity shocks are *i.i.d.* and denoted by $\varepsilon = \{\varepsilon_L, \varepsilon_H\}$ with $z = 1$ for all the firms. The probability of the low shock is π_L and the probability for the high shock is $\pi_H = 1 - \pi_L$. We study how the

firms' decisions vary as a function of the level of long debt it owes. With only temporary uncertainty that is *i.i.d.* over time, the set of long debt contracts offered to firms is constant over time and across firms. In particular the available set is independent of the level of level of long debt B_R , productivity realization ε and short-term contract (K, K_R) .

Lemma 1. Default sets are increasing in long debt.

If $\varepsilon \in D(B_R^1, K, K_R)$, then $\varepsilon \in D(B_R^2, K, K_R) \forall B_R^2 > B_R^1$.

Firms default when they hold high long debt. This is because both current dividend $d(B_R, K, K_R, \varepsilon)$ and the contract value $W^c(B_R, K, K_R, \varepsilon)$ are decreasing in long debt while the default value is independent of long debt.

Lemma 2. Long debt contracts are bounded. There exists an $\bar{B} > 0$ such that $B' \leq \bar{B}$ for all (B', B'_R) contracts.

The set of long debt contracts are bounded by the maximum set up costs that a project can have while remaining profitable. In an environment without perfect enforcement, the entrepreneur will only participate in the project if the expected value delivers a positive value. The project at most will generate the expected lifetime discounted profit under the optimal investment K_{fb} , $M = \left[E(\varepsilon)K_{fb}^\alpha - K_{fb} \right] / r$. Thus projects with larger set up costs will for sure not be financed because they can not deliver expected positive value to both the lender and the entrepreneur. Hence, the set of long contracts every period offered to firms that remain in operations are bounded such that $\bar{B} \leq M$.

Moreover, the value of each project in the economy with imperfect enforcement will be lower than in an economy with perfect enforcement due to the added constraints. We show below that in our model capital depends on the level of long debt and firms will be inefficiently small. This implies that the maximum set up cost for which projects will be financed are strictly smaller in our economy relative to the first best $\bar{B} < M$. In addition, if with this

maximum loan of \bar{B} default sets are non-empty the corresponding long contract (\bar{B}, \bar{B}_R) will imply that $\bar{B}_R \geq \bar{B}(1+r)$.

Definition. *Debt Overhang.* A firm with long debt B_R faces ‘debt overhang’ if when it invests the first best capital stock, it cannot satisfy the non-negative dividend conditions for all shock realizations.

Proposition 1. Debt overhang exists. There exists a B^* such that for all $B_R > B^*$, $\varepsilon_L K_{fb}^\alpha - K_{Rfb} - B_R + B' < 0 \forall (B', B'_R)$.

Define $B^* = \bar{B} + \varepsilon_L K_{fb}^\alpha - K_{Rfb}$. For any $B_R > B^*$, the non-negative dividend condition for ε_L cannot be satisfied. The first best capital stock is chosen based on average productivity. However the non-negative dividend condition must be satisfied for every shock realization. Thus if ex-post the low shock is realized for some firm the first best capital stock is too big because the short-term loan is on an inefficiently big capital stock. Given that enforcement problems limit the resources long debt contracts provide, a firm with large long debt that invests the first best capital and receives the low productivity shock will have negative dividends.

Note that in our model firms can have debt overhang for $B_R < \bar{B}$ if $\bar{B} + \varepsilon_L K_{fb}^\alpha - K_{Rfb} < \bar{B}_R$. Given that $\bar{B}_R \geq \bar{B}(1+r)$ the necessary condition is that $\varepsilon_L K_{fb}^\alpha - K_{Rfb} < r\bar{B}$. We can then choose a particular distribution of shocks to guarantee this condition for any $r\bar{B} \geq 0$.

Proposition 2. Default sets are the lower set.

If $\varepsilon_2 \in D(B_R, K, K_R)$, then $\varepsilon_1 \in D(B_R, K, K_R)$ for any $\varepsilon_1 < \varepsilon_2$.

Proof. See Appendix.

Firms default when productivity is low. If default is involuntary then it is clear that default happens for low shocks because current dividends $d(B_R, K, K_R, \varepsilon)$ are decreasing in ε . If default is voluntary, then firms default for low productivity because the contract value $W^c(B_R, K, K_R, \varepsilon)$ is increasing in productivity at a faster rate than the default value, which is constant across the temporary productivity shock. The reason is that the set of long debt contracts the firm can choose under the low shock, such that the non-negative dividend condition is satisfied, are also feasible under the high shock.

Proposition 3. A firm with debt overhangs under-invests relative to the first best.

Proof. See Appendix.

A firm that faces ‘debt overhang’ has two choices: (1) the firm chooses not to default for both shocks, but adjusts the investment such that the non-negative dividend condition is satisfied for ε_L ; (2) the firm chooses to default for some shock and adjusts the investment decision according to default choices. The proposition shows both cases lead to underinvestment relative to the first best.

A firm that faces debt overhang and decides to repay debt, cannot have the first best capital because this would lead to having negative dividends. Thus the firm decreases its investment such that profits under ε_L are high enough to deliver a non-negative dividend. This necessarily leads to underinvestment relative to the first best because smaller sizes of firms increase ex-post profits for ε_L .

The firm with debt overhang could also decide to default for some shock realizations. If the firm chooses to default, this will happen only for the case of low productivity ε_L . This is because default sets are the lower set due to proposition 2, and because they are less than the whole set when projects produce positive output.

Capital choices are based on maximizing expected profits and relaxing the non-negative dividend condition ex-post state by state in the repayment states. Regarding the maximization of profits firms choose capital such that the marginal product equals the marginal cost. When firms choose to default under some shocks, the expected marginal product of capital is lower

than if firms do not default because default states involve direct costs as the firm loses output in the low shock. However the expected marginal cost of capital is the same regardless of the default decisions because short-term lenders break even. These two features push the optimal capital stock to be smaller than the first best when default sets are non-empty. Regarding relaxing the non-negative dividend condition, this is only relevant for the choice of capital if it is binding under ε_H .³ However, under ε_H the non-negative dividend condition is relaxed by also decreasing the capital stock. The reason is that given that the short-term contract carries a default premium and the capital that maximizes $\varepsilon_H K^\alpha - (1+r)/\pi_H K$ is smaller than K_{fb} . Thus if the firm wants to use capital to relax the non-negative dividend condition in the high shock, this also implies that it must under-invest relative to the first best. It is key that the short-term contract adjusts according to default choices. If the short-term contract would not be adjusted to default choices a defaulter firm could find it optimal to over-invest relative to the first best.

Thus firms with high levels of long-term debt will for sure under-invest regardless of their default choices. The next session explores quantitatively this magnitude of the inefficient scale in production.

4.2 Numerical Examples

This section presents some numerical examples to illustrate the underinvestment feature in our model under temporary shocks alone (set $z = 1$). Table 2 presents the parameters used in these examples. The detailed calibration of these parameters will be presented in the next section.

Table 2. Parameter Values

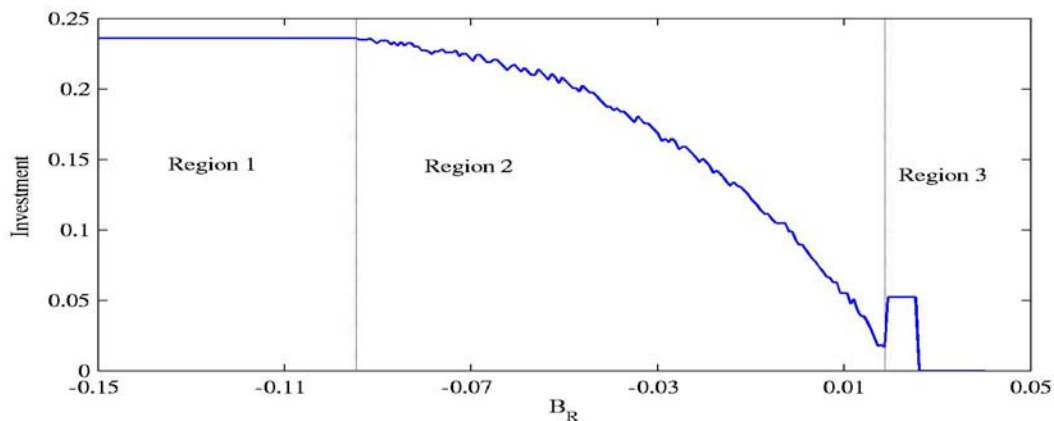
Value

³ The non-negative dividend condition under z_L is irrelevant if default is chosen here.

Set up costs I	0.007	
Re-draw probability θ	0.30	
Discount factor β	0.90	
Decreasing returns α	0.90	Atkeson and Kehoe 2005
Interest rates r	0.04	US annual interest rate
Death rate δ	0.05	Exit rate of large firms
Shock process for example	$\varepsilon_L = 0.7, \varepsilon_H = 1.08$	
with temporary shocks	$\pi_L = 0.2, \pi_H = 0.8$	

Given the above parameterization, we first analyze the firm's decision rules as a function of long debt levels, and then present the limiting distribution of firms. Based on understanding of decision rules and the limiting distribution, we analyze the time series dynamics of firms using simulated examples.

Figure 2 plots investment, output and dividend of a firm as a function of its long debt normalized by aggregate GDP which is constant. The upper panel presents investment decisions, which are independent of shock realizations given the nature of incomplete markets and temporary shocks. The middle and the lower panel plot output and dividend conditional on receiving a high or a low productivity shock.



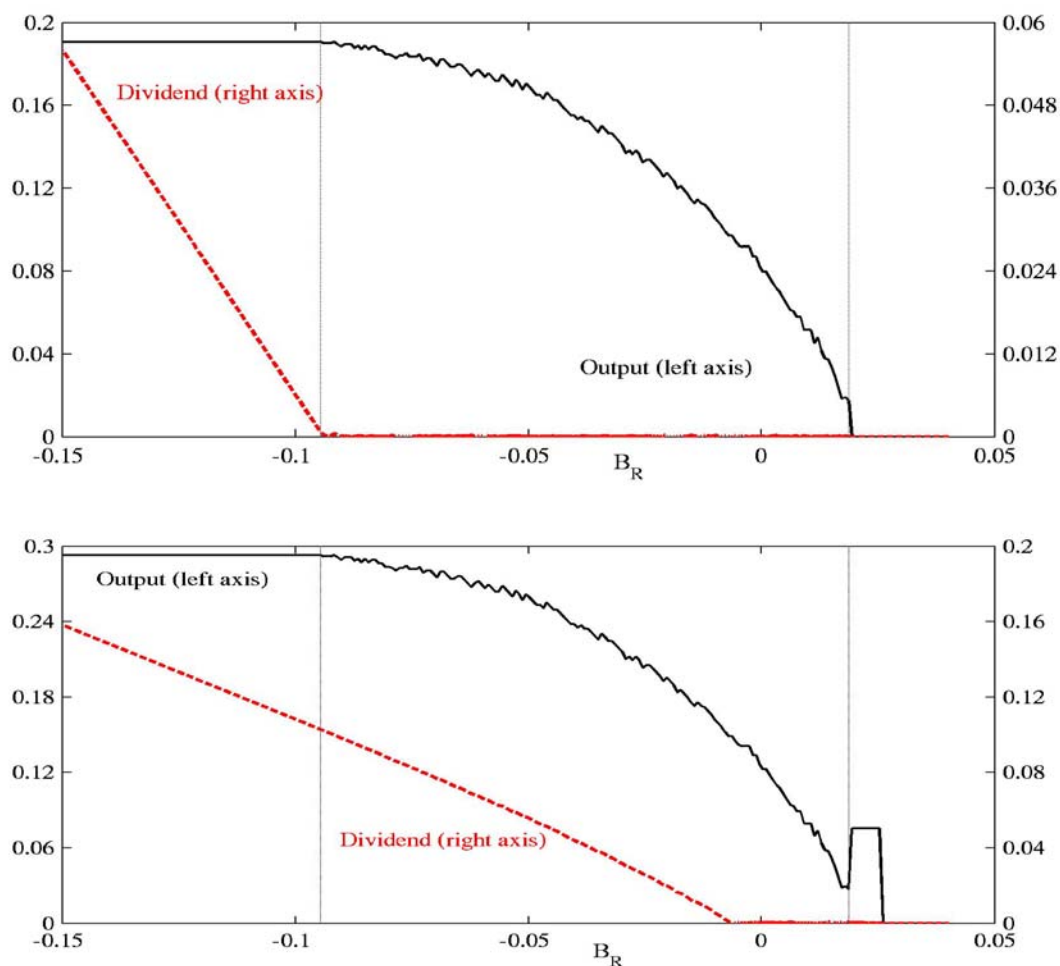


Figure 2. Firm's investment, output and dividend as a function of long debt holdings

We can classify 4 distinct regions across long debt over GDP ratios according to investment decisions. In the first region long debt is small enough such investment is equal to the first best. The second region corresponds to the case where the firm faces a binding non-negative dividend condition when the low shock is realized. Here the firm underinvests as prescribed in Proposition 3 to avoid default by increasing profits in the low state such that the non-negative dividend condition is satisfied. The reason that the firm is willing to be small is that the continuation value of keeping the project $W(B')$ is very large once the initial setup costs have been already paid.

When long debt is big enough, the firm finds it optimal to default if the low shock is realized; this is region 3. In anticipation of such an event the firm under-invests relative to the

first best as in Proposition 3, but increases investment relative to region 2. The reason for the increase in investment is that given that in the low shock the short loan will not be repaid, the firm finds it optimal to adjust its size for a more appropriate scale in the high shock. However given the more expensive short contracts, this increase in investment is never big enough such that capital is higher than in the first best. For the highest debt levels in region 3, the firm can further under-invests because the non-negative dividend condition binds for the high shock, and decreasing investment relaxes this constraint.

Finally in region 4, long debt is so high, that for any positive output the firm will find it optimal to default. Thus in equilibrium short lenders are unwilling to provide the firm with any contracts that would give positive output. An alternative way to think about region 4 is that projects with the initial set up costs corresponding to region 4 will not be financed in equilibrium. Note though that under perfect enforcement some of these projects would be financed because the value of the firm under perfect enforcement is uniformly higher than under imperfect enforcement for all initial set up costs. This is because the value of the firm is lower due to underinvestment when debt is high.

Our model shows that in the economy with the enforcement friction and incomplete contracts, firms with high debt will be inefficiently small. One main reason behind the result is that the set of short contracts available for these firms are very costly relative to firms with less debt. This can be shown in Figure 3, which plots the short contract price schedules for firms with two different debt levels. Clearly, the short contract price schedule for high B is uniformly lower than that for low B .

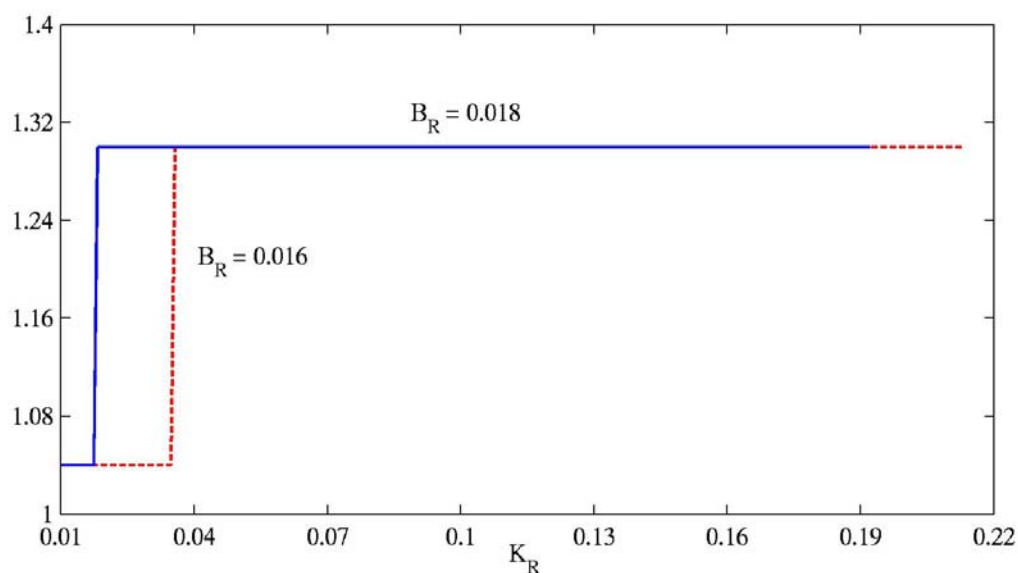


Figure 3. Interest Rate Premium for Different Debt Levels

Given the parameters of the model and the above decision rules, the model generates a limiting distribution of firms, shown in Figure 4. In the limiting distribution, a substantial portion of firms hold large long debt positions, and thus the firm size distribution contains many inefficiently small firms. (The first best investment level is 0.24.) Over 70% of firms have an investment level almost two thirds of the first best. Some firms have a very small size, or about 10 times smaller than in a frictionless world.

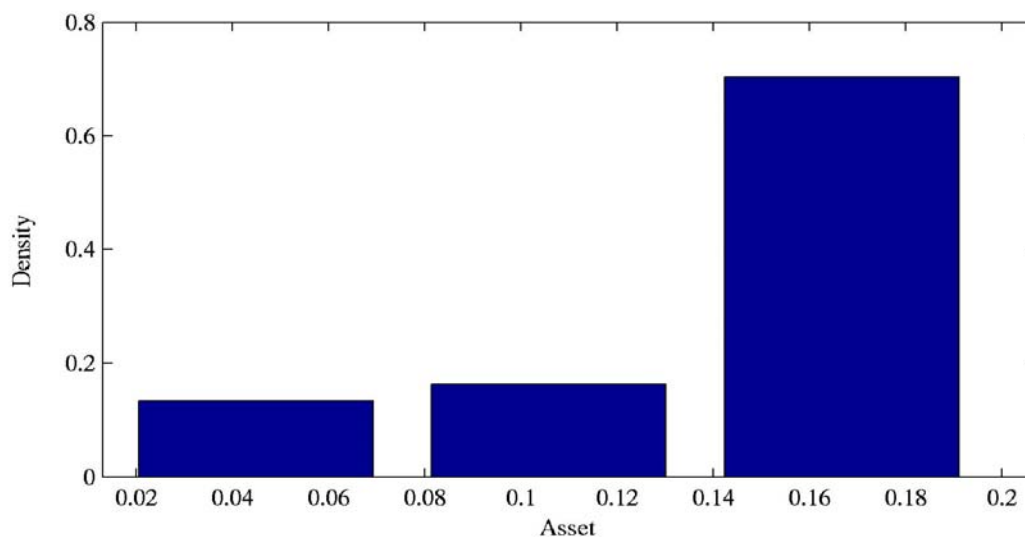
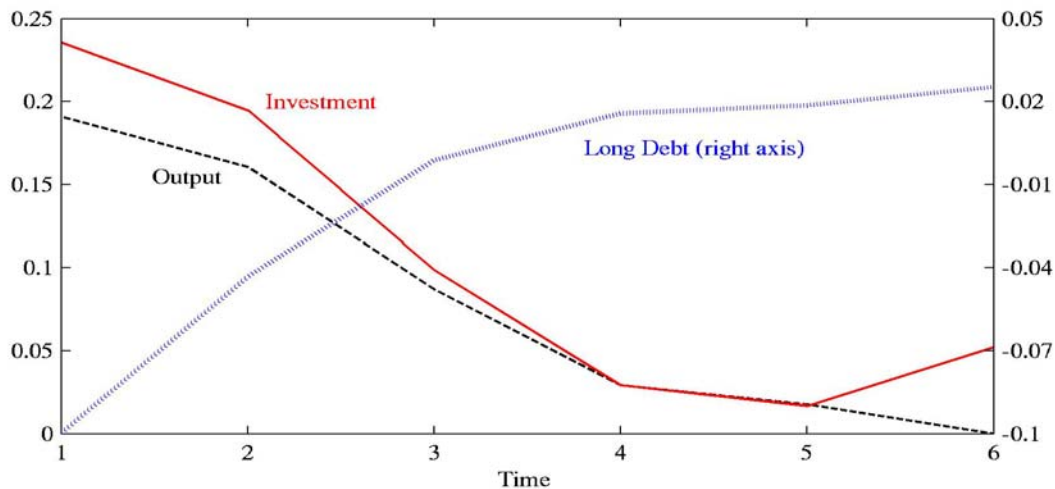


Figure 4. Firm Size Distribution

The result that enforcement problems induce firms to be inefficiently small is also present in models of enforcement frictions with a complete set of assets (Quintin (2000), Cooley, Marimon and Quadrini (2004), and Clementi and Hopenhayn (2006)). Thus our added feature of incomplete markets is unnecessary for qualitatively getting this result. However, our key difference is that incomplete markets amplify the inefficiency in scale for some firms because the value of the firm can decrease over time due to higher long debt holdings. Enforcement problems are more severe when the value of the firm is low, thus as firms increase debt holdings the enforcement friction limits further their ability to produce efficiently. For a given initial start up cost, lack of enforcement plus incomplete markets generates a larger portion of inefficiently small firms in the invariant distribution.

We now present some time series examples. The time series dynamics of a firm's investment and output after a series of shocks is driven by the dynamics of long debt, as short-term investment contracts do not carry any persistent effects in our model.⁴ Figure 5 presents the time series dynamics for two firms: the upper panel plots output and long-term bond dynamics for a firm that experiences a sequence of only low productivity shocks; the lower panel plots the same statistics for a firm that experiences a sequence of only high productivity shocks. The initial debt position was chosen such that firms start in region 2.



⁴ This is because we have assumed that depreciation of capital equals 1. If depreciation was to be lower, capital dynamics would interact with long debt dynamics.

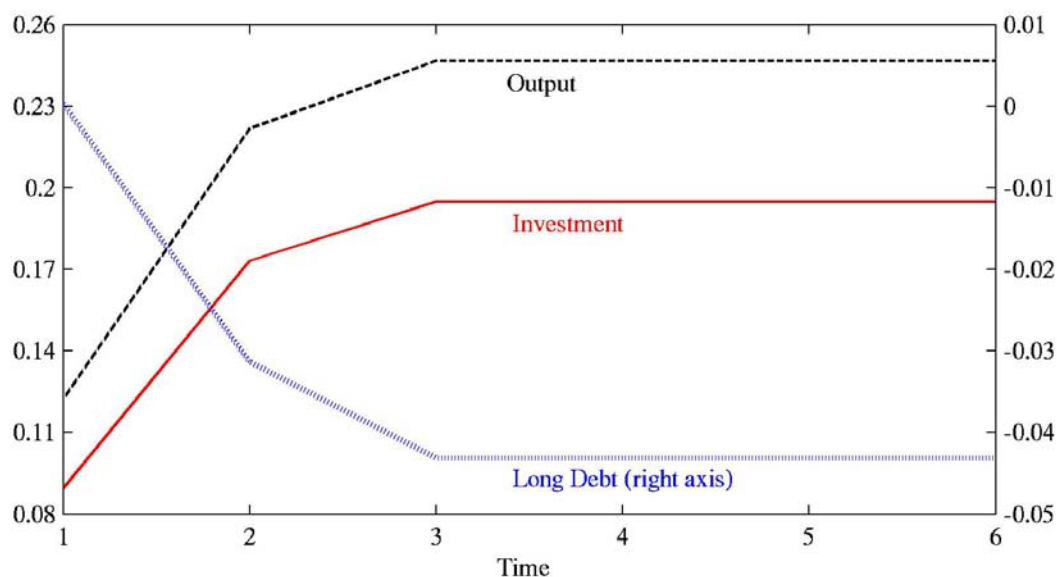


Figure 5. Dynamics of firm's output, capital and long debt

Let's first consider the case of a firm that faces low productivity shocks. This firm produces less and less output while it increases its long debt holdings over time. The firm chooses to remain in operation because of better future prospects, but it produces very little contemporaneously. After a long enough sequence of low shocks, 6 periods in this simulation, long debt grows so much, such that the firm finds it optimal to go bankrupt and exit. Anticipating a default event in period 6, the capital invested increases because the firm does not repay back the short loan in the low shock and thus dividends are strictly positive.

Incomplete markets are essential for enforcement frictions to amplify bad productivity shocks over time. Our interpretation is that enforcement problems are exacerbated for firms that have a history of low productivity. A sequence of low shocks lowers the value of the firm because uncontingent long debt has to grow over time to cover interest payments of an increasingly higher stock of debt. This feature differentiates our model from existing models that look at the implications of enforcement problems under a complete set of assets. In those models the value of the firm cannot go down ever because state contingent long debt allows the firm to efficiently pay off the set up costs only when high shocks are realized. By introducing incomplete markets our model is able to maintain the underinvestment feature of those models, while producing the additional feature of amplification for low shocks

Let’s now consider the case of a firm who faces a sequence of good productivity shocks as in the lower panel of Figure 5. A firm who initially has a high initial long debt, starts operating at an inefficient scale. However if the firm receives a high productivity shock then it is able to repay enough of its long debt, such that in the next period it becomes unconstrained. This shows that our model also generates amplification of good shocks because after a sequence of high shocks the value of the firm increases enough such that it becomes unconstrained. Our model shares the ‘positive persistence’ feature of models with enforcement frictions and a complete set of assets have.

3.2 Statistics of Firms Dynamics

Now we look at the relations between exit, growth and leverage with the firm size, which are reported in Table 3. We rank a large number of firms from the limiting distribution according to their size (investment) and put them into three bins with an equal number of firms. We next report for each bin average statistic of firm dynamics. First, the model generates the size-growth relationship as in the data; smaller firms have higher growth rates. Conditional upon survival (under the high shock), small firms will decrease their debt next period, which implies a much larger investment next period. This leads to a high growth rate relative to the large less-constrained firms. Second, the model matches the data in that small firms have higher probability of exit (or lower probability of survival) because they face a greater default probability. Third, the model predicts that exit firms have higher leverage ratios than non-exit firms, which is also consistent with the data. This is because exit firms are mostly small firms with high debt, which gives rise to higher leverage ratio than non-exit firms. However, with the temporary productivity shocks alone, the model generates a negative relationship between leverage and size, which is opposite in the data.

Table 3: Statistics of Firm Dynamics with Temporary Shocks Only

	Growth	Exit	Leverage	Leverage of Exit Firms
Bin 1	1.55	5.51%	1.04	1.06

Bin 2	0.93	5.00%	0.80	0.80
Bin 3	0.91	5.00%	0.79	0.79

Note: Leverage is defined as $(K_R + B_R)/(K + I)$ in the model.

Our model matches the data in that small firms pay less dividends and their investment is sensitive to cash flow. Furthermore, our model has several empirical testable implications in terms of the dynamics of firms. First it predicts that firms with high debt will have more volatile output conditional on a given volatility of shocks. Second, our model predicts that the history of productivity matters for a long time in terms of firms' output. In particular even under temporary shocks as in this example, firms can become smaller and smaller over time because they face very more restrictive financial contracts after a history of low productivity.

4. Firms dynamics with permanent productivity differences

This section studies the implications from lack of enforcement in contracts for the leverage and size relation when productivity differences across firms are only permanent. Our main result is that permanently more productive firms have higher degree of enforcement and can sustain more debt relative to their larger assets without having incentives to default. The reason is that the value of a more productive firm is bigger relative to a constant value of re-drawing a new project that might not have the high permanent productivity. The surplus from the relation with lenders is larger for these firms not only because of larger assets but also because of larger effective degree of enforcement.

We solve the economy for three permanent productivity bins: $\{z_L, z_M, z_H\}$ and no idiosyncratic shocks. The values chosen for z $z_L = .95$, $z_M = 0.98$, $z_H = 1.0$. The other parameters are as in table 2. Figure 6 plots the repayment probabilities for each type of firm with z permanent productivity as a function of the debt relative to assets B_R / K . Without uncertainty the repayment probabilities jump from one to zero at threshold level of debt. As the figure shows

the level of debt relative to assets B_R / K for which default probabilities are one, is larger for the z_H project than for the z_L project. Given that the size of the project K is increasing in z , this implies that the maximum absolute amount of sustainable debt is larger for the larger scale projects.

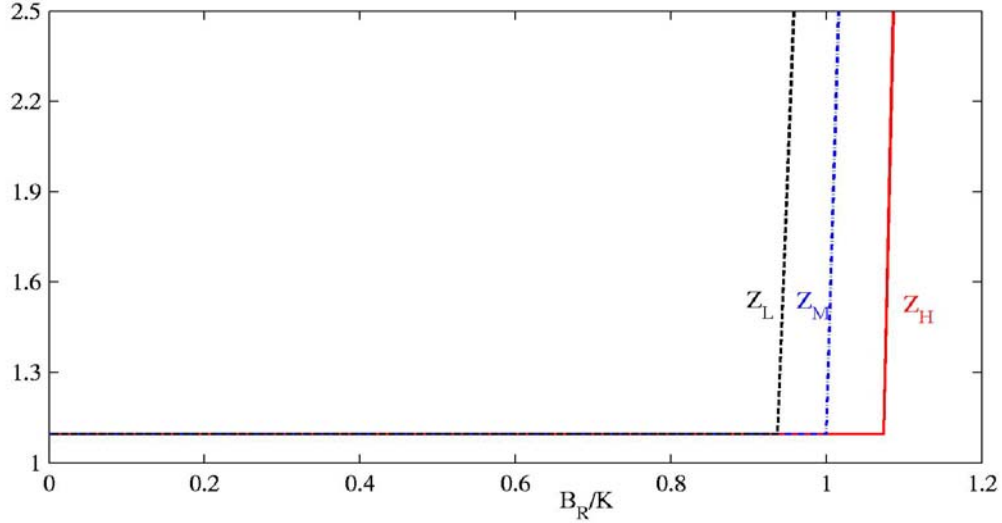


Figure 6. Interest Rate Premium and Leverage

Table 4 presents the leverage ratios that in equilibrium firms with z productivity chose. The table shows that enforcement frictions can rationalize the empirical fact that larger firms have larger leverage ratios when the size of firms varies due to permanent productivity differences. High productivity firms have larger scales in K , yet they can borrow more debt relative to their larger assets because the surplus from honoring contracts with lenders is larger given that there is more at stake in event of default.

Table 4. Size and Leverage with permanent productivity differences

Productivity	Size (K)	Leverage ($(B_R + K_R)/(K + I)$)
z_L	0.95	1.88
z_M	0.98	1.96

z_H

1.00

2.05

5. Quantitative Implications of the Model

We now present results for the full model with both transitory and permanent. We want to assess quantitatively our mechanism in reproducing the facts regarding exit, growth, leverage and size for the dataset of firms in Ecuador.

Calibrate the permanent and transitory productivity process to match certain features of the firm data in Ecuador. Table 5 reports the parameters values used for the shocks. All other parameters are as in Table 2.

We simulate the model over 500 periods for 3000 firms. At every point in time there is a cross section distribution of firms that we use to compute the model statistics. Given that we don't have aggregate uncertainty, the model delivers a stationary distribution of firms over debt and assets. We divide at every point in time the cross section of firms into 3 bins based on size K such that an equal number of firms is in each bin. We then compute for every bin and for the whole distribution the average leverage, leverage of the firms that exit, exit rate, and output growth. Table 6 reports averages of these statistics across the last 100 periods.

Table 5. Calibrated Parameter Values

	Value	Target
Permanent productivity	$z_L = 0.95, z_M = 0.98, z_H = 1.0$	Size distribution
	$\pi_{z_L} = 0.2, \pi_{z_M} = 0.5, \pi_{z_H} = 0.3$	$K_{\max} / K_{\min} = 9$
Temporary productivity	$\sigma_{\varepsilon_{zL}} = 0.15$	Range in sales variance
	$\sigma_{\varepsilon_{zM}} = 0.20$	0.30–0.45
	$\sigma_{\varepsilon_{zH}} = 0.25$	

Firms in the third bin in the distribution are firms with high permanent productivity and who have not received a history of bad idiosyncratic shocks. Firms in the second bin correspond to firms in the middle permanent productivity category who have not experienced a history of bad shocks. Firms with z_H who have received a short history of low shocks are also in bin two. Bin one firms are the z_L firms together with z_M and z_H firms that have received long histories of adverse shocks.

We first report the statistics for the overall firm distribution and then for the 3-quantile classification of firms according to size. Firms on average grow by 17%, exit at a rate of 5.4%, and have on average 0.8 as a leverage ratio. The leverage ratio of the exiting firms is higher on average and equal to 0.85. Small firms grow faster than large firms as in the case with only temporary productivity differences. Many firms in the first bin are small because they have received a sequence of bad shocks, and thus they are very sensitive to good shocks because they can reduce their debt holdings and increase their capital stock towards a better size. In the general model the exit-size relation is also the one found in the data, although the exit rates of small firms is lower than in the data. In the first bin, small firms with a sequence of low productivity realizations are prone to exit if they receive another low shock because their project becomes is not longer profitable given the large debt they owe.

Table 6. Firm Dynamics and Size

	Size	Growth	Exit	Leverage	Leverage of Exit
Overall	0.11	1.17	5.4	0.80	0.85
First 3-quantile	0.04	1.60	5.76	0.99	1.09
Second 3-quantile	0.12	0.99	5.00	0.67	0.67
Third 3-quantile	0.17	0.91	5.00	0.74	0.74

The leverage size relation in the model is non-monotonic, because of the two distinct debt usages. Small firms in the first bin with permanently low productivity do not have large debt levels, but those with relatively higher productivity that are small because of a history of

unlucky draws are precisely small due to their large debt holdings. This effect pushes upward the average leverage of firms in bin one. However this effect allows us to generate that firms who exit have a larger debt to asset ratios as in the data as shown in the last column of the table. The leverage-size ranking for the second and third bin are in the empirically correct ranking. Large firms have larger leverages than medium size firms because they have larger degree of enforcement as in the case of only permanent productivity differences.

Finally figure 7 plots the firm size distribution for the case of permanent and transitory productivity differences. Permanent productivity differences across firms allow the model to better the firm size distribution of assets.

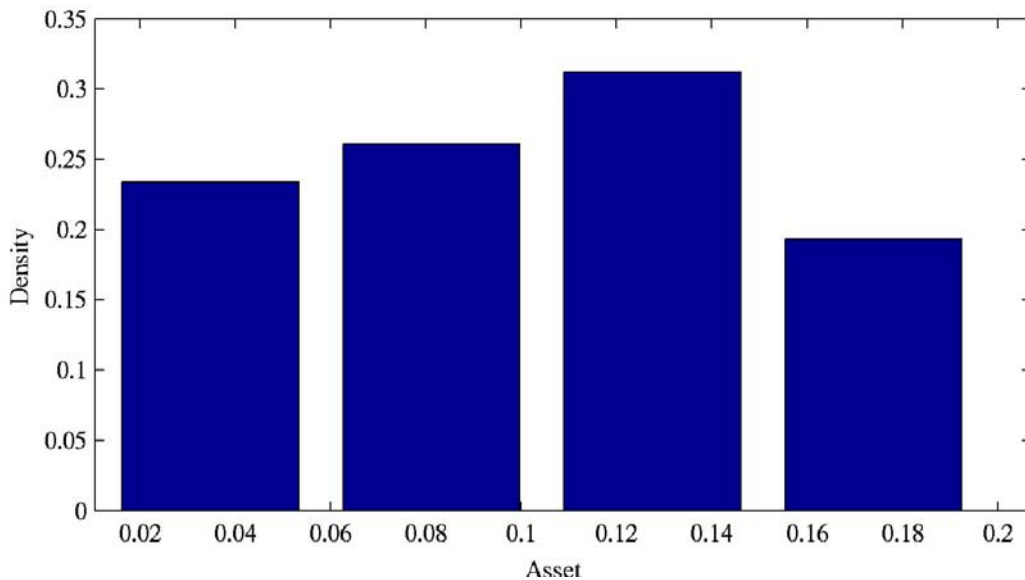


Figure 7. Asset Distribution in the Model

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Appendix 1

Proposition 2. Default sets are the lower set.

If $z_2 \in D(B, K, P)$, then $z_1 \in D(B, K, P)$ for any $z_1 < z_2$.

Proof: We need to show that for any $z_1 < z_2$, $W^c(B, K, P, z_1) < \eta z_1 K^\alpha$, i.e.,

$(1 - \eta)z_1 K^\alpha - P - B + L_1^* + \beta W^o(B_1'^*) < 0$, where $(L_1, B_1'^*)$ is the optimal long debt contract.

From $z_2 \in D(B, K, P)$, we have $(1 - \eta)z_2 K^\alpha - P - B + L_2^* + \beta W^o(B_2'^*) < 0$, where $(L_2, B_2'^*)$ is

the optimal long debt contract. Define the feasible set of long debt contract (L, B') under

$W^c(B, K, P, z)$ as $\Gamma(K, P, B, z) = \{(L, B') \mid zK^\alpha - P - B + L \geq 0\}$. Clearly

$\Gamma(K, P, B, z_1) \subset \Gamma(K, P, B, z_2)$. Thus, we have

$$0 > (1 - \eta)z_2 K^\alpha - P - B + L_2^* + \beta W^o(B_2'^*) \geq (1 - \eta)z_2 K^\alpha - P - B + L_1^* + \beta W^o(B_1'^*)$$

$$\geq (1 - \eta)z_1 K^\alpha - P - B + L_1^* + \beta W^o(B_1'^*). \text{ Q.E.D.}$$

Proposition 3. A firm with debt overhangs under-invests relative to the first best.

Proof: A firm with debt overhangs has two choices: either repays under both shocks or defaults under the low shock, but repays under the high shock. We show that the firm will under-invest relative to the first best under either choice.

Choice 1: To repay debt under both shocks, the firm has to adjust investment away from the first best to guarantee the non-negative dividend (NND) condition under z_L . That is, the firm has to increase the profit under z_L , only by lowering investment from K_{fb} .

Choice 2: We first show that K_{fb} is dominated by some investment level smaller than the first best, and then show that K_{fb} dominates any investment higher than the first best.

(1) Underinvestment: the first best investment K_{fb} is dominated by some K_a , smaller than

K_{fb} , i.e., $K_a = K_{fb} - \varepsilon < K_{fb}$. We need to show that the expected welfare under K_a , denoted by $A(B, K_a, P_a)$ is higher than that under K_{fb} , denoted by $A(B, K_{fb}, P_{fb})$, where the expected welfare is defined as $A(B, K, P) = E(\max\{W^c(B, z, K, P), \eta z K^\alpha\})$.

Assume that the short contracts are designed according to defaulting only for the low shock. We can find an $\varepsilon_0 > 0$ such that choosing $K_a > K_{fb} - \varepsilon_0$ increases short profits under both shocks and the expected profit relative to choosing K_{fb} , where ε_0 is defined as

$\varepsilon_0 = K_{fb} - (\alpha(\pi_H z_H + \eta \pi_L z_L) / (1+r))^{1/(1-\alpha)} > 0$. Consider two cases when the firm chooses K_a : defaults under z_L and repays under both shocks.

(a) When choosing K_a , the firm defaults under z_L .

In this case, we know $A(B, K_a, P_a) = \pi_H W^c(B, z_H, K_a, P_a) + \eta \pi_L z_L K_a^\alpha$ and

$A(B, K_{fb}, P_{fb}) = \pi_H W^c(B, z_H, K_{fb}, P_{fb}) + \eta \pi_L z_L K_{fb}^\alpha$. Assume that $(L^*(K_{fb}), B^*(K_{fb}))$ is the optimal long contract of $W^c(B, z_H, K_{fb}, P_{fb})$. Since the profit under K_a is higher than that under K_{fb} when z_H occurs, $L^*(K_{fb})$ is feasible under $W^c(B, z_H, K_a, P_a)$. Thus, we have

$$W^c(B, z_H, K_a, P_a) \geq z_H K_a^\alpha - (1+r)K_a / \pi_H - B + L^*(K_{fb}) + \beta \delta W^o(B^*(K_{fb})).$$

To show $A(B, K_a, P_a) > A(B, K_{fb}, P_{fb})$, it is suffice to show

$$(\pi_H z_H + \pi_L \eta z_L) K_a^\alpha - (1+r)K_a > K_{fb}^\alpha - (\pi_H z_H + \pi_L \eta z_L)(1+r)K_{fb}, \quad (14)$$

which is guaranteed by our choice of K_a .

(b) When choosing K_a , the firm repays for both shocks.

In this case, we have $A(B, K_a, P_a) = \pi_H W^c(B, z_H, K_a, P_a) + \pi_L W^c(B, z_L, K_a, P_a)$.

We know that $A(B, K_a, P_a) \geq \pi_H W^c(B, z_H, K_a, P_a) + \pi_L \eta z_L K_a^\alpha$ because the firm chooses to repay under the low shock. As in case (a), we have

$$\begin{aligned} W^c(B, z_H, K_a, P_a) &\geq z_H K_a^\alpha - (1+r)K_a - B + L^*(K_{fb}) + \beta \delta W^o(B^*(K_{fb})) \\ &\geq z_H K_a^\alpha - (1+r)K_a / \pi_H - B + L^*(K_{fb}) + \beta \delta W^o(B^*(K_{fb})). \end{aligned}$$

Thus, to show $A(B, K_a, P_a) > A(B, K_{fb}, P_{fb})$ is equivalent to show inequality (14), which is true for our choice of K_a .

(2) No overinvestment: any investment greater than the first best K_{fb} is dominated by K_{fb} .

The profits under both shocks and also the expected profit will decrease when the firm over-invests. Follows the similar argument above, we could show that any investment greater than the first best K_{fb} is dominated by K_{fb} . Q.E.D.

Appendix 2: Computational Algorithm

When solving the model numerically, we follow the following computation algorithm:

(1) Given the world interest rate R , we start the initial guess of long and short contracts as $P = (1+r)K$ and $B' = (1+r)L$.

(2) Given the long and short contracts, value functions and decision rules are solved through the value function iterations: $\{W(B, z_{-1}), W^o(B, K, P, z), W^c(B, K, P, z), V^d(K, z), K(B, z_{-1}), P(B, z_{-1}), L(B, K, P, z), B'(B, K, P, z), d(B, K, P, z), A(B, K, P), D(B, K, P)\}$.

- Start with an initial guess of the value function $W_j(B, z_{-1})$.
- Compute value functions $V^d(K, z)$ and $W^c(B, K, P, z)$ with optimal long contract $(L(B, K, P, z), B'(B, K, P, z))$. Set $W^c(B, K, P, z) = -\infty$ if the set of the feasible allocation is empty.
- Compare $V^d(K, z)$ with $W^c(B, K, P, z)$ and get the default decision $d(B, K, P, z)$. Thus, we get the repayment set $A(B, K, P)$, the default set $D(B, K, P)$ and the value function $W^o(B, K, P, z)$.
- Update the value function $W_{j+1}(B, z_{-1})$ by choosing the optimal short contract $(K(B, z_{-1}), P(B, z_{-1}))$.
- Iterate the above procedures until W converges.

(3) Given the decision rules and value functions, we can updated the long and short contracts according to

$$K = \frac{P}{(1+r)} \left(1 - \int_{D(B, K, P)} dG(z, z_{-1}) \right) \text{ and } L = \frac{B'}{1+r} \left(1 - \int_{D(B', K'(B', z), P'(B', z))} dG(z', z) \right).$$

(4) Iterate (1)-(3) until the short and long contracts converges.