Idiosyncratic Investment Risk
in a Neoclassical Growth Economy*

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Abstract
This paper examines the effects of idiosyncratic investment, entrepreneurial, or capital-income risk in a neoclassical growth economy, with heterogeneous infinitely-lived risk-averse households, competitive labor and product markets, and incomplete risk sharing. I consider both the case that all capital in the economy is private equity and the case that there is also public equity. I characterize the general equilibrium in closed form under standard assumptions for preferences and technologies. The presence of a risk premium on private investment leads to a lower capital stock in the steady state as compared to complete markets. The interest rate is also lower, but need not be monotonic with respect to the degree of risk sharing. In calibrated versions of the model, the reduction in the steady-state level of output is in the order of 15%. Finally, the interaction of risk premia and wealth introduces a novel propagation mechanism.

Key Words: incomplete markets, entrepreneurial risk, investment, growth, fluctuations, propagation, amplification.

JEL Classification Numbers: E5, E6.

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1 Introduction

A large portion of investment and production takes place in privately-held businesses, which implies that investors are often exposed to large undiversifiable idiosyncratic risks in their investment returns and their capital income. Even in a developed economy like the United States, privately-held business account for a big portion of production, employment, and capital. In the 70’s and 80’s, private equity in the United States was about twice as large as public equity; at the pick of the bull market (late 90’s), privately-held businesses still accounted for almost half of corporate equity. The probability that a privately-held firm survives over the first 5 years of its life is only 37 percent. Even conditional on survival, the variation in private investment returns appears extreme. Moskowitz and Vissing-Jørgensen (2002) further document the “dramatic lack of diversification” in the portfolios of entrepreneurs and private-equity investors. More that three-fourths of private equity is owned by households for whom private equity constitutes at least half of their total net worth; and households who own private equity invest on average more than 70 percent of their private equity holdings in a single company in which the household has an active management interest. Similarly, Carrol (2001) documents that the median rich household holds more than half his financial wealth in private equity. What is more, idiosyncratic capital-income risk is not limited to private equity; lack of diversification is evident in public-equity holdings as well (e.g., Guvenen, 2002).

This paper examines the implications of introducing idiosyncratic investment and capital-income risk in the neoclassical growth framework. The economy is populated by a large number of heterogeneous infinitely-lived risk-averse households and a large number of perfectly competitive firms. Each firm is owned by only one household, and each household owns only one firm. Firms produce a single homogeneous good, using the capital stock invested by the household that owns the firm, and employing labor in the competitive labor market. Production exhibits constant returns to scale (CRS) with respect to capital and labor and diminishing marginal returns with respect to the capital-labor ratio. Households are infinitely lived and have constant relative risk aversion (CRRA) and constant elasticity of intertemporal substitution (CEIS). They earn labor income by supplying their labor in the competitive labor market and capital income by investing in the firm they own. Production and firm earnings are subject to firm-specific productivity or demand shocks,
which translate to idiosyncratic investment and capital-income risk for the households. Households can freely borrow and lend in a riskless bond and therefore face no credit constraints in their consumption and investment choices; but they can not trade any other financial assets and therefore can not diversify their idiosyncratic investment and capital-income risk.

At the aggregate level, the economy exhibits diminishing returns to capital accumulation, like in the standard complete-markets neoclassical growth model. At the individual level, however, firms earnings are linear in private investment, as each firm operates under CRS and can keep its employment level proportional to its capital stock for any given wage rate and any given realization of firm-specific productivity. The individual household’s decision problem thus reduces to a non-stationary homothetic problem similar to the classic portfolio problem studied by Samuelson (1969) and Merton (1969). The difference is that investment risk is idiosyncratic rather than aggregate and that investment returns, risk premia, and labor income are all endogenous in general equilibrium. Nevertheless, the optimal decision rules are linear in individual wealth for given prices, which ensures that the mean of the wealth distribution is a sufficient statistic for aggregate dynamics. I am thus able to obtain closed-form solution for the general equilibrium of an incomplete-markets infinite-horizon growth economy under standard assumptions for preferences and technology, which, to the best of my knowledge, is a methodological innovation.

The presence of uninsurable idiosyncratic capital-income risk introduces a risk premium on private investment, which reduces the demand for capital and increases the demand for the riskless bond at any level of the interest rate and any level of wealth. In equilibrium, the interest rate adjusts so that the excess aggregate demand for the riskless bond is zero. Incomplete risk sharing thus leads to a lower interest rate, which tends to offset the adverse effect of the private risk premium on the demand for investment. In other words, the demand for investment falls because of risk aversion, but the supply of savings rises because of the precautionary motive. It follows that the effect of incomplete markets on capital accumulation is generally ambiguous. However, unless the elasticity of intertemporal substitution is implausibly low, in which case a reduction in interest rates leads to a large increase in saving rates, the effect of higher risk premia dominates the effect of lower interest rates. As a result, the steady-state levels of capital, output, and wages are lower as compared to complete markets. In calibrated versions of the model, the reduction in the steady-state level of output is around 18%. Note that these results contrast with the increase
of capital predicted by traditional Bewley models (Aiyagari, 1994; Krusell and Smith, 1998).

The above result turns out to be robust to whether all capital in the economy is in risky private equity or part of it is in riskless public equity. In the presence of public equity, agents can mitigate the effect of an increase in the idiosyncratic risk associated with private equity on their consumption by rebalancing their savings towards public equity. However, both private equity and public equity are held in equilibrium, it must be that private equity is on average more productive than public equity. It follows that a worsening in risk sharing opportunities reduces aggregate productivity by shifting resources from private production to public equity. At the same time, a worsening in risk sharing opportunities may have an ambiguous effect on savings and the interest rate. In calibrated versions of the model, the mitigating effect of public equity is quite strong as regards aggregate savings, but rather moderate as regards productivity and output. In the benchmark calibration, the reduction in the steady-state level of output is around 15% in the presence of public equity as compared to 18% in the absence of public equity.

Turning to transitional dynamics, I find that the interaction between wealth, risk taking, and investment generates a novel propagation mechanism. When risk taking depends positively on wealth (as in the case of CRRA or, more generally, diminishing absolute risk aversion), the anticipation of low economic activity and low income in the future feeds back to high risk premia and low investment demand in the present. But if all agents scale down their investment, low economic activity in the near future becomes a partially self-fulfilling prophecy. This kind of dynamic macroeconomic complementarity can slow down convergence to the steady state and can generate amplification and persistence over the business cycle.

This effect is most striking when the economy is small and open to an international market for the riskless bond, in which case the risk-free rate is exogenously given. Under complete risk sharing, the capital stock would be pinned down by the equality of the marginal product of capital with the international interest rate, implying that domestic investment is independent of either domestic income or domestic savings. Moreover, convergence to the steady state would be instantaneous, unless one introduces adjustment costs. In the presence of undiversifiable investment risk, however, risk premia are sensitive to wealth, implying that domestic investment can be highly correlated with domestic income and domestic savings. Moreover, convergence to the steady state can take long time (even in the absence of adjustment costs) and any exogenous income shock is amplified
through the interaction of risk taking and wealth.\footnote{The implied correlation between domestic investment and domestic savings perhaps provides a partial resolution to the Feldstein-Horioka puzzle. Also, the international allocation of capital and the world distribution of income are determined, not merely by technologies, but also by tastes and the degree of risk sharing within each country.}

The rest of the paper is organized as follows. Section 2 relates the paper to the pertinent literature. Section 3 introduces the model. Section 4 characterizes individual behavior. Section 5 derives the general equilibrium in closed form and analyzes the steady state. Section 6 extends the model to two sectors (private and public equity). Section 7 discusses the propagation mechanism and Section 8 concludes. All proofs are presented in the Appendix.

\section{Related Literature}

...to be completed...

\section{The Model}

Time is discrete, indexed by $t \in \{0, 1, ..., \infty\}$. The economy is populated by a continuum of infinitely-lived households, indexed by $i$ and distributed uniformly over $[0, 1]$. Each household owns a single firm, and each firm is owned by a single household. I thus use the index $i$ interchangeably for households and firms, with the understanding that firm $i$ is the firm owned by household $i$. Firms produce a single homogeneous good under constant returns to scale with respect to capital and labor. Firm $i$ must use the capital stock accumulated by household $i$ but may employ labor in a competitive labor market. Household $i$ is endowed with $N^i$ units of effective labor, which it supplies inelastically in the competitive labor market. I let an arbitrary distribution for $N^i$ and normalize its cross-sectional mean to one. Each household can invest in its own firm, but in no other firm, and production is subject to firm-specific productivity risk. Households can not diversify the idiosyncratic risk in their capital income (firm earnings), but can freely trade on a riskless bond. The economy is open to an international market of the riskless bond, which exogenously fixes the interest rate. Finally, idiosyncratic shocks wash out at the aggregate level, ensuring that aggregate dynamics are deterministic.
3.1 Firms, Technology, and Idiosyncratic Risks

In the beginning of every period \( t \), nature draws a random variable \( A_i^t \) for each household \( i \). \( A_i^t \) is independently and identically distributed across \( i \) and \( t \), with compact support \( A \subset \mathbb{R}^+ \), differentiable c.d.f. \( \Psi \), and mean normalized to one. The random variable \( A_i^t \) represents an exogenous productivity (or demand) shock specific to the firm (or the capital stock) that household \( i \) owns.

The gross output that this firm produces in period \( t \) (including the non-depreciated portion of the installed capital) is given by

\[
y_i^t = F(k_i^t, n_i^t, A_i^t),
\]

where \( k_i^t \) denotes the capital stock that household \( i \) has accumulated by the beginning of period \( t \), \( n_i^t \) denotes the amount of labor that the firm hires during period \( t \), and \( F : \mathbb{R}^3_+ \to \mathbb{R}^+ \) is a neoclassical production technology. \( F(K, L, A) \) is twice differential in all its arguments; it has constant returns to scale with respect to \( K \) and \( L \); the marginal products of either \( K \) or \( L \) are positive and strictly diminishing; and the appropriate Inada conditions are satisfied. The function \( F \) is common across agents, but different agents receive different realizations of \( A_i^t \). There is no exogenous uncertainty other than the idiosyncratic shocks \( A_i^t \), which are observed in the the beginning of period \( t \). This generates a filtration \( \{\mathcal{F}_t\}_{t=0}^{\infty} \), where \( \mathcal{F}_t \) denotes the information set in period \( t \). \( k_i^t \) is \( \mathcal{F}_{t-1} \)-measurable, but \( n_i^t \) is \( \mathcal{F}_t \)-measurable. That is, investment is chosen before the realization of idiosyncratic production risk, but employment can be adjusted ex post.

In order to interpret a higher \( A \) as higher productivity, higher demand, or higher profitability, I impose \( F_A > 0 \), \( F_{KA} > 0 \), and \( F_{LA} > 0 \). I let 0 be the minimum of the support of \( A \) and assume \( F(K, L, 0) = 0 \), meaning that the worst idiosyncratic event leads to zero production. It follows that \( F(K, L, A) > 0 \) for all \((K, L, A) > 0 \). I normalize \( A \equiv E A = 1 \). All the analytical results are derived for a general specification of the idiosyncratic risk. At some points, however, it will prove useful to focus on the following case:

Assumption A1 \( F(K, L, A) = F(AK, L, 1) \) and \( \ln A \sim \mathcal{N}(-\sigma^2/2, \sigma^2) \). That is, the idiosyncratic shock is augmented to capital and lognormally distributed.

Note that \( \sigma \) then parsimoniously parametrizes the degree of incomplete risk sharing.

The net earnings of firm \( i \) are given by

\[
\pi_i^t \equiv y_i^t - \omega_t n_i^t = F(k_i^t, n_i^t, A_i^t) - \omega_t n_i^t, \tag{1}
\]
where $\omega_t$ denotes the competitive wage rate in period $t$. Note that firm earnings $\pi^i_t$ represent capital income for household $i$; firm-specific production risk thus translates to household-specific capital-income risk.

### 3.2 Households, Budget, and Preferences

The budget constraint of household $i$ in period $t$ is

$$c^i_t + k^i_{t+1} + b^i_{t+1} = \pi^i_t + R_t b^i_t + \omega_t N^i.$$  

(2)

On the expenditure side, $c^i_t$ is consumption, $k^i_{t+1}$ is investment in physical capital, and $b^i_{t+1}$ is savings in risk-free bonds. On the income side, $\pi^i_t$ is capital income, given by (1), $\omega_t N^i$ is labor income, and $R_t b^i_t$ is the value of the bond portfolio. $R_t$ denotes the gross risk-free rate between period $t - 1$ and period $t$, while $\omega_t$ denotes the wage rate in period $t$.

Naturally, consumption and physical capital can not be negative: $c^i_t \geq 0$ and $k^i_t \geq 0$. Households can freely borrow in the riskless bond up to the natural solvency constraint that debt is low enough to be paid out by household income even under the worst realization of idiosyncratic uncertainty. Given that the worst realization of idiosyncratic uncertainty corresponds to $\pi^i_t = 0$, the natural solvency constraint is given by

$$R_t b^i_t \geq - (\omega_t N^i + h^i_t),$$  

(3)

where

$$h^i_t \equiv \sum_{\tau=1}^{\infty} \frac{w_t + \tau N^i}{R_{t+1}...R_{t+\tau}}$$

is the present value of future labor income.

Turning to preferences, it is useful to distinguish between intertemporal substitution and risk aversion. I thus adopt a Kreps-Porteus/Epstein-Zin preference specification. A stochastic consumption stream $\{c^i_t\}_{t=0}^{\infty}$ generates a stochastic utility stream $\{u^i_t\}_{t=0}^{\infty}$ defined by the recursion

$$u^i_t = U(c^i_t) + \beta \cdot U \{ \mathbb{CE}_t[U^{-1}(u^i_{t+1})] \},$$  

(4)

for all $t \geq 0$. The quantity $\mathbb{CE}_t(u) \equiv \Upsilon^{-1}[\mathbb{E}_t \Upsilon(u)]$ denotes the certainty equivalent of $u$; the utility functions $U$ and $\Upsilon$ aggregate consumption across dates and states, respectively. Finally, as
standard in growth and finance, I assume constant elasticity of intertemporal substitution (CEIS) and constant relative risk aversion (CRRA):

\[ U(c) = \frac{c^{1-\theta}}{1 - 1/\theta} \quad \text{and} \quad \Upsilon(c) = \frac{c^{1-\gamma}}{1 - \gamma}. \quad (5) \]

\( \theta > 0 \) is the elasticity of intertemporal substitution and \( \gamma > 0 \) is the degree of relative risk aversion.\(^2\)

Note that this specification nests standard expected utility as a special case: When \( \theta = 1/\gamma \), (4) reduces to standard expected utility, \( u_t = \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j (c_{t+j})^{1-\gamma} / (1 - \gamma) \).

3.3 Equilibrium

I let \( W_t, C_t, K_t, B_t, N_t, \) and \( \Pi_t \) denote the cross-sectional averages of \( w_i, c_i, k_i, b_i, n_i \), and \( \pi_i \) and I define an equilibrium (with deterministic aggregates) as follows:

**Definition 1** A competitive equilibrium is a sequence of prices \( \{\omega_t, R_t\}_{t=0}^{\infty} \) and a collection of contingent plans \( \{c_i, n_i, k_{i+1}, b_{i+1}\}_{t=0}^{\infty}, i \in [0, 1] \), such that:

(i) For every household \( i \), and given prices \( \{\omega_t, R_t\}_{t=0}^{\infty} \), the plan \( \{c_i, n_i, k_{i+1}, b_{i+1}\}_{t=0}^{\infty} \) maximizes life-time utility (4), subject to the budget constraint (2), the restriction that \( (c_i, n_i, k_{i+1}, b_{i+1}) \) be \( \mathcal{F}_t \)-measurable, and the non-negativity restriction that \( (c_i, n_i, k_{i+1}, w_{i+1}) \geq 0 \), for all \( t \).

(ii) The labor market clears in every \( t \); that is, \( \int n_i = 1 \).

(iv) The credit market clears in every \( t \); that is, \( \int b_i = 0 \).

4 Optimal Individual Behavior

This section characterized optimal individual behavior, taking equilibrium prices as given.

4.1 Labor Demand and Capital Income

Consider how firm \( i \) chooses labor demand \( n_i \) in period \( t \). Observe that labor demand \( n_i \) affects only entrepreneurial income \( \pi_i \) in period \( t \). Further, \( n_i \) is chosen after the capital stock \( k_i \) has been sunk in the firm and after the contemporaneous productivity shock \( A_i \) has been observed. It follows that labor demand \( n_i \) is optimal if and only if it maximizes \( \pi_i \) for any given \( k_i \) and \( A_i \).

\(^2\)To be precise, specification (4) is well defined if and only if \( \theta, \gamma \neq 1 \); standard adjustments can be made to nest the logarithmic case.
Moreover, because production exhibits constant returns to scale, the optimal \( n^i_t \) and \( \pi^i_t \) turn out to be proportional to \( k^i_t \).

**Proposition 1** Given \((\omega_t, A^i_t, k^i_t)\), optimal labor demand \( n^i_t \) and capital income \( \pi^i_t \) are linear in \( k^i_t \), decreasing in \( \omega_t \), and increasing in \( A^i_t \):

\[
n^i_t = n(A^i_t, \omega_t)k^i_t \quad \text{and} \quad \pi^i_t = r(A^i_t, \omega_t)k^i_t,
\]

where \( r(A, \omega) \equiv \max_L [F(1, L, A) - \omega L] \) and \( n(A, \omega) \equiv \arg \max_L [F(1, L, A) - \omega L] \).

**Lemma 1** Under A1, optimal labor demand \( n^i_t \) and capital income \( \pi^i_t \) are linear in \( A^i_t \) as well. That is, \( n(A, \omega) = A\pi(\omega) \) and \( r(A, \omega) = A\pi(\omega) \), where \( \pi(\omega) \equiv \int n(A, \omega)d\Psi(A) \) and \( \pi(\omega) \equiv \int r(A, \omega)d\Psi(A) \).

### 4.2 Consumption, Savings, and Investment

Let \( w^i_t \equiv \pi^i_t + R_t b^i_t + \omega_t N^i \) denote the household’s financial wealth, or cash-in-hand. The sum \( w^i_t + h^i_t \) can be interpreted as “effective” wealth and the (3) is equivalent to imposing \( w^i_t + h^i_t \geq 0 \) for all realizations of \( A^i_t \). Following Proposition 1, financial wealth reduces to \( w^i_t = r(A^i_t, \omega_t)k^i_t + R_t b^i_t + \omega_t N^i \).

For any given deterministic price sequence \( \{\omega_t, R_t\}_{t=0}^{\infty} \) and labor endowment \( N^i \), let \( V^i_t(w) \) denote the value function and \((c^i_t(w), k^i_{t+1}(w), b^i_{t+1}(w))\) the optimal policy rules, as functions of financial wealth. The Bellman equation is

\[
V^i_t(w^i_t) = \max_{(c^i_t, k^i_{t+1}, b^i_{t+1})} U(c^i_t) + \beta \cdot UT^{-1} \{ \mathbb{E}_t [YU^{-1}V^i_{t+1}(w^i_{t+1})] \}
\]

s.t. \[
c^i_t + k^i_{t+1} + b^i_{t+1} = w^i_t,
\]

\[
w^i_{t+1} = r(A^i_{t+1}, \omega_{t+1})k^i_{t+1} + R_t b^i_{t+1} + \omega_{t+1}N^i,
\]

\[
c^i_t \geq 0, \quad k^i_{t+1} \geq 0, \quad w^i_{t+1} + h^i_{t+1} \geq 0.
\]

This problem is formally similar to the classic portfolio problem studied by Samuelson (1969), Merton (1969), and Bodie et al (1992). In particular, preferences are homothetic (by assumption), financial wealth is linear in the all types of investment (by implication of Proposition 1), and labor income is risk-free (provided that wage rate is deterministic). These properties permit closed-form solution and ensure that the optimal decision rules are linear in effective wealth.
It is useful to define the functions $\rho(\omega, R) \equiv \max \phi \left\{ \mathbb{E} [\phi r(A_i, \omega) + (1 - \phi)R]^{1-\gamma} \right\}^{1/(1-\gamma)}$ and $\phi(\omega, R) \equiv \arg \max \phi \left\{ \mathbb{E} [\phi r(A_i, \omega) + (1 - \phi)R]^{1-\gamma} \right\}^{1/(1-\gamma)}$. We can then show:

**Proposition 2** The equilibrium plan $\{w^i_t, c^i_t, b^i_{t+1}, k^i_{t+1}\}_{t=0}^{\infty}$ for household $i$ satisfies

\[
\begin{align*}
    w^i_t &= r(A^i_t, \omega^i_t)k^i_t + R^i_t b^i_t + \omega^i_t N^i,
    c^i_t &= (1 - s^i_t)(w^i_t + h^i_t),
    k^i_{t+1} &= s^i_t \phi^i_t (w^i_t + h^i_t),
    b^i_{t+1} &= s^i_t (1 - \phi^i_t)(w^i_t + h^i_t) - h^i_t,
\end{align*}
\]

where $s^i_t = \left[ 1 + \left( \sum_{\tau=t}^{\infty} \prod_{j=t}^{\tau} \beta \theta \rho^j_{\theta - 1} \right)^{-1} \right]^{-1}$, $\rho_t = \rho(\omega_{t+1}, R_{t+1})$, and $\phi_t = \phi(\omega_{t+1}, R_{t+1})$.

Individual wealth, consumption, savings, and investment are stochastic, but the sequence $\{h^i_t, s^i_t, \rho^i_t, \phi^i_t\}_{t=0}^{\infty}$ is deterministic as long as the price sequence $\{\omega_t, R_t\}_{t=0}^{\infty}$ is deterministic. $s^i_t$ is the saving rate out of effective wealth and $\phi^i_t$ is the fraction of savings allocated to capital. The quantity $\rho^i_t$, on the other hand, represents the the risk-adjusted return to savings (aka, the certainty equivalent of the overall portfolio return). The optimal share $\phi^i_t$ maximizes this risk-adjusted return. The saving rate is an increasing (decreasing) function of current and future risk-adjusted returns if and only if $\theta > 1$ ($\theta < 1$, respectively).

There is no exact closed-form solution for the optimal $\rho^i_t$ and $\phi^i_t$, but we can derive an approximate closed-form solution following Campbell and Viceira (2002). This proves particularly useful in the case that the idiosyncratic productivity shock is augmented to capital.

**Lemma 2** Under $A1$,

\[
\ln \rho_t \approx \ln R_{t+1} + \frac{[\ln \tau(\omega_{t+1}) - \ln R_{t+1}]^2}{2\gamma \sigma^2} \quad \text{and} \quad \phi_t \approx \frac{\ln \tau(\omega_{t+1}) - \ln R_{t+1}}{\gamma \sigma^2}.
\]

(8)

It follows that both the optimal share $\phi_t$ of savings allocated to productive capital and the resulting risk adjusted return on savings decreases with an increase in idiosyncratic volatility $\sigma$ or an increase in the anticipated wage rate $\omega_{t+1}$. On the other hand, an increase in the risk free rate $R_{t+1}$ lowers $\phi_t$ but raises $\rho_t$. 
5 General Equilibrium

5.1 Closed-Form Solution

Proposition 1 established that optimal labor demand $n_i^j$ and capital income $\pi_i^j$ for any individual $i$ are linear in her capital stock $k_i^j$ and decreasing in the wage rate $\omega_t$. By this property and the i.i.d. assumption for $A_i^j$, it follows that $N_t = \bar{\pi}(\omega_t)K_t$ and $\Pi_t = \bar{\pi}(\omega_t)K_t$, where $\bar{\pi}(\omega) \equiv \int n(A, \omega)d\Psi(A)$ and $\bar{\pi}(\omega) \equiv \int r(A, \omega)d\Psi(A)$. That is, aggregate employment and aggregate capital income are linear in the aggregate capital stock and decreasing in the wage rate.³ Define the functions $\omega(K) \equiv \pi^{-1}(1/K)$ and $f(K) \equiv \pi(\omega(K))K + \omega(K)$, and note that, in equilibrium, $\omega_t = \omega(K_t)$ and $\Pi_t + \omega_t = f(K_t)$. The quantity $f(K_t)$ measures domestic income plus non-depreciated capital;⁴ net national income is given by $Y_t = f(K_t) - K_t + (R - 1)B_t$. We can now characterize the general equilibrium in closed form:

**Proposition 3** The equilibrium path $\{C_t, K_t, Y_t, H_t, s_t, \omega_t, R_t\}_{t=0}^\infty$ satisfies

\[
C_t + K_{t+1} = Y_t = f(K_t),
\]

\[
C_t = (1 - s_t)[Y_t + H_t],
\]

\[
K_{t+1} = \phi_t s_t [Y_t + H_t],
\]

\[
H_t = \frac{1}{R_{t+1}}[\omega_{t+1} + H_{t+1}],
\]

\[
(1 - s_t) = \frac{1}{1 + \beta R_{t+1}^\rho (1 - s_{t+1})^{-1}},
\]

\[
\bar{\pi}(\omega_t)K_t = 1,
\]

where $\phi_t = \phi(\omega_{t+1}, R_{t+1})$ and $\rho_t = \rho(\omega_{t+1}, R_{t+1})$.

Condition (9) gives the resource constraint of the economy. Conditions (10) and (11) give, respectively, the aggregate consumption level and the aggregate capital stock, as shares of effective

³Because $n(A, \omega)$ and $r(A, \omega)$ are monotonic in $A$, a mean preserving spread in $A$ increases the cross-sectional variance of both labor demand and capital income, whatever the technology. The consequent effect on mean labor demand $\bar{\pi}(\omega)$ and mean capital returns $\bar{\pi}(\omega)$ depends on the curvature of $n(A, \omega)$ and $r(A, \omega)$ with respect to $A$. When $A$ is augmented to $K$, the latter are linear in $A$, implying that a mean-preserving spread in $A$ leaves $\bar{\pi}(\omega)$ and $\bar{\pi}(\omega)$ unaffected. The effect of TFP and labor-augmented shocks, on the other hand, is ambiguous.

⁴Under A1, it can be shown that $f(K) = F(K, 1, \overline{A})$. More generally, however, $f(K)$ may differ from $F(K, 1, \overline{A})$. 
wealth. (12) and (13) give, respectively, the present value of aggregate labor income and the optimal saving rate, in recursive form. Finally, (14) is the labor market clearing condition.

5.2 Steady State

A steady state is a fixed point of the dynamic system (??)-(??). Since the general equilibrium was characterized in closed form for any kind of idiosyncratic productivity risk, so does the steady state as well. For expositional simplicity, however, it is most useful to consider the case that the productivity shock is augmented to capital, which I assume for the rest of the paper. We can then show

**Proposition 4** In steady state, the capital stock $K$ and the ratio of capital to effective wealth $\phi$ solve

$$
E \left\{ [\phi f'(K) + (1-\phi)R]^{-\gamma} [Af'(K) - R] \right\} = 0,
$$

$$
\beta^\theta \left\{ E \left[ [\phi f'(K) + (1-\phi)R]^{1-\gamma} \right] \right\}^{\theta-1} \times \left[ \phi f'(K) + (1-\phi)R \right] = 1,
$$

while the interest rate $R$ solves

$$
\frac{f(K) - f'(K)K}{(R-1)K} = \frac{1-\phi}{\phi}.
$$

The wage rate and the level of consumption are then given by $\pi(\omega)K = 1$ and $C = f(K) - K$.

Condition (15) is simply the orthogonality condition that characterizes the optimal allocation of savings between risky capital and riskless bonds. Condition (16), which follows from the resource constraint and the Euler condition, also has a simple interpretation. The first term in the left-hand side of (16) is the steady-state value of the saving rate: $s = \beta^\theta \rho^{\theta-1}$, where $\rho = \left\{ E \left[ [\phi f'(K) + (1-\phi)R]^{1-\gamma} \right] \right\}^{1/(1-\gamma)}$ measures the risk-adjusted return to savings. The second term is the aggregate return to savings. The product of the two terms gives the growth rate of savings. In the steady state, savings must be constant, which gives (16). Finally, condition (17)

---

5 Although aggregates are well defined at the steady state, the wealth distribution degenerates as $t \to \infty$. This is a common feature of various incomplete-markets models (e.g., Lucas, 1992) and can be fixed, for example, with the following modification. For arbitrary $\lambda \in (0, 1)$, assume that, in every period $t$, a randomly selected mass $\lambda$ of households dies and is replaced with a mass $\lambda$ of new households; and the sum of the assets of the exiting households is distributed uniformly among the entering households. There is then a non-degenerate stationary wealth distribution.
follows from the bond market clearing condition and means that the ratio of the present value of labor income to the capital stock must equal \((1 - \phi)/\phi\).

When markets are complete, (15) collapses to \(f'(K) = R\). That is, the marginal product of capital equals the risk-free rate. When instead markets are incomplete, any solution to (15) and (16) must satisfy \(f'(K) > R\), with the gap between the mean return to investment and the interest rate representing the risk premium on private equity.

After some tedious algebra, it can be shown that, for any given level of the interest rate, conditions (15) and (16) determine a unique steady-state level of capital, which is decreasing in the amount of idiosyncratic risk.

**Proposition 5** Under A1, the steady state satisfies

\[
\ln f'(K) \approx \ln R + \sigma \sqrt{\frac{2\gamma \theta}{1 + \theta}} [-\ln(\beta R)].
\]

If the economy were open to an international market for the riskless bond, the interest rate would be exogenously fixed, the steady-state level of domestic capital would be a decreasing function of the amount of idiosyncratic risk, and the current account would absorb any residual between domestic savings and domestic investment. In this case, financial innovation would unambiguously stimulate investment and production in the economy, while it may have an ambiguous effect on the current account.

In a closed economy, however, the interest rate must adjust so that the aggregate excess demand for the riskless bond is zero, which is what condition (17) imposes. Incomplete risk sharing now implies also a reduction in the interest rate, which counteracts with the increase in the risk premium on private investment. The overall effect on the capital stock is thus ambiguous in general. To determine which effect is more likely to dominate, I ran an extensive series of numerical simulations. The effect of a higher risk premium turned out to dominate the effect of a lower interest rate unless savings were strongly negatively related to real returns, that is, unless the elasticity of intertemporal

---

\(^6\)Economies with macroeconomic complementarities often exhibit multiple steady states when the complementarity is sufficiently strong. Such multiplicity appears to be absent in the present model. Nonetheless, it may well arise in an OLG variant of the model.

\(^7\)The steady-state levels of foreign asset holding \((B_\infty)\) and consumption \((C_\infty)\) are also uniquely determined. This result contrasts with complete markets, in which case a steady state requires \(R = 1/\beta\) but leaves the levels of foreign-assets holdings and consumption indeterminate \((B_\infty \text{ and } C_\infty \text{ are increasing functions of initial } B_0)\).
substitution were implausibly low. In the next subsection, I present some representative numerical results.

5.3 Numerical Simulations

The parameter $\sigma$ corresponds to the coefficient of variation in private investment returns. To see this, note that, under A1, $\text{Var}_{t-1}[r(A_{t+1}, \omega_t)] = \sigma^2\bar{r}(\omega_t)^2 = \left\{\sigma \mathbb{E}_{t-1}[r(A_{t+1}, \omega_t)]\right\}^2$. Unfortunately, accurate estimates of $\sigma$ are not readily available. Campbell et al. (2001) look at the idiosyncratic risk in publicly-traded stocks. To the extent that this risk is a good proxy for the risk in privately-held businesses, their evidence suggest a value for $\sigma$ well above 100%. Moskowitz and Vissing-Jørgensen (2002) find that the returns to a value-weighted index of private equity funds have a standard deviation of 17% as compared to a mean of 14%. Since the index diversifies away much of the firm-specific risk, that evidence also suggests a value for $\sigma$ well above 100%. To be conservative, I consider both $\sigma = 50\%$ and $\sigma = 100\%$. The rest of the parameters are calibrated as usually. In the benchmark calibration, the time period is one year, the discount rate is $1 - \beta^{-1} = 5\%$, the coefficient of relative risk aversion is $\gamma = 2$, the elasticity of intertemporal substitution is $\theta = 1$, the production function is Cobb-Douglas with income share of capital $\alpha = 40\%$, and the depreciation rate is $\delta = 10\%$.

The results for the benchmark calibration are presented in the first two rows of Table 1. The rest of the Table does a series of robustness checks for different values for $\gamma, \theta, \beta, \alpha,$ and $\delta$.

...to be completed...

[insert Table 1 here]

6 Two Sectors: Private and Public Equity

I now introduce a second sector, in which production is risk free. This sector is meant to capture public equity. Households can now invest in publicly tradeable capital as well as in privately held capital; they continue to trade riskless bonds and to receive labor income as in the benchmark model.
Let $X_t$ and $L_t$ denote the aggregate capital and labor allocated to the public-equity sector and $G$ the aggregate production function of that sector. To simplify the analysis, it is useful to assume that

**Assumption A2** $G(X, L) = F(X, L, \bar{A}/\mu)$ for some $\mu > 1$. That is, the technology in the public-equity sector is identical to that in the private-equity sector, except that the mean productivity of the latter is higher than the mean productivity of the former.

Depending on the productivity difference $\mu$ and the level of risk in private equity, investment and production in either sector may be zero in equilibrium. If $\mu \leq 1$, for example, public equity would dominate private equity. If instead $\mu > 1$, the private-equity sector is necessarily active, but the public-equity sector may or may not be active. In what follows, I focus on the case that both sectors are active; I later discuss under what conditions public equity may become obsolete.

### 6.1 General Equilibrium

Consider the decision problem of the individual household. Since publicly-traded capital is risk-free, simple arbitrage implies that the return to public equity equals the return to the riskless bond. Letting $x^i_t$ denote the stock of publicly-traded capital held by household $i$, we now write his budget constraint as

$$c^i_t + k^i_{t+1} + x^i_{t+1} + b^i_{t+1} \leq r(A^i_t, \omega_t)k^i_t + R_t x^i_t + R_t b^i_t + \omega_t N^i.$$  

It follows that the equilibrium plan for household $i$ satisfies

$$w^i_t = r(A^i_t, \omega_t)k^i_t + R_t x^i_t + R_t b^i_t + \omega_t N^i,$$

$$c^i_t = (1 - s_t)(w^i_t + h^i_t),$$

$$k^i_{t+1} = s_t \phi_t(w^i_t + h^i_t),$$

$$x^i_{t+1} + b^i_{t+1} = s_t(1 - \phi_t)(w^i_t + h^i_t) - h^i_t,$$

where $\phi_t$, $\rho_t$, and $s_t$ are defined as before.

Consider next the general equilibrium. Define $l(\omega) \equiv \arg\max_l[G(1, l) - \omega l]$ and $R(\omega) \equiv \max_l[G(1, l) - \omega l]$ and note that both $l(\omega)$ and $R(\omega)$ are decreasing in $\omega$. By profit maximization, $\omega_t = G_{\omega}(X_t, L_t)$ and $R_t = G_X(X_t, L_t)$, or equivalently $L_t = l(\omega_t)X_t$ and $R_t = R(\omega_t)$; and by constant returns to scale, $R(\omega_t)X_t + \omega_t L_t = G(X_t, L_t)$. Similarly, $N_t = \pi(\omega_t)K_t$ and $\pi(\omega_t)K_t + \omega_t N_t =$
Finally, Assumptions A1 and A2 imply $\pi(\omega_t) = \mu(\omega)$ and $\pi(\omega) = \mu R(\omega)$. The latter means that the parameter $\mu$ pins down the private equity premium whenever both sectors are active and, by implication, the optimal $\phi$ solves $E\{[\phi A\mu + 1 - \phi]^{-\gamma} [A\mu - 1]\} = 0$ and is thus determined by $\mu$ and $\sigma$ alone. Using these results, aggregating across $i$, and following similar steps as in Proposition 3, we conclude that

**Proposition 6** Suppose A1 and A2 hold. In any equilibrium in which both sectors are active, the equilibrium dynamics satisfy

$$C_t + K_{t+1} + X_{t+1} = Y_t = F(K_t, \pi(\omega_t) K_t, \overline{A}) + G(X_t, l(\omega_t) X_t),$$

$$C_t = (1 - s_t) [Y_t + H_t],$$

$$K_{t+1} = \phi_t s_t [Y_t + H_t],$$

$$H_t = \frac{1}{R_{t+1}} [\omega_{t+1} + H_{t+1}],$$

$$\pi(\omega_t) K_t + l(\omega_t) X_t = 1,$$

$$R_t = R(\omega_t),$$

$$(1 - s_t) = \frac{1}{1 + \beta^\theta (\rho_t)^{\theta - 1} (1 - s_{t+1})^{-1}},$$

where $\phi_t = \phi$, $\rho_t = \phi R(\omega_{t+1})$, $\varrho \equiv \left\{ E[\phi A\mu + 1 - \phi]^{1 - \gamma} \right\}^{1/(1 - \gamma)}$, and $\phi$ is the unique solution to $E\{[\phi A\mu + 1 - \phi]^{-\gamma} [A\mu - 1]\} = 0$.

### 6.2 Steady State

A steady state in which both sectors are active is a fixed point of the dynamic system. The following proposition characterizes this steady state in closed form.

**Proposition 7** Suppose A1 and A2 hold. A steady state in which both sectors are active is unique whenever it exists, and it necessarily exists for sufficiently high $\sigma$ or sufficiently low $\mu$. The steady-state interest rate is then given by

$$(\beta R)^\theta \varrho ^{-1} (\phi \mu + 1 - \phi) = 1,$$

where $\varrho = \left\{ E[\phi A\mu + 1 - \phi]^{1 - \gamma} \right\}^{1/(1 - \gamma)}$ and $\phi$ solves $E\{[\phi A\mu + 1 - \phi]^{-\gamma} [A\mu - 1]\} = 0$. The steady-state wage rate is then given by $R(\omega) = R$ and the steady-state levels of private and public
Idiosyncratic Investment Risk

It is now straightforward to consider the comparative statics of the steady state with respect to the degree of risk sharing.

**Proposition 8** There exists $\theta < 1$ such that, whenever $\theta > \theta$, an increase in idiosyncratic risk ($\sigma$) raises the interest rate ($R$) and has an ambiguous effect on aggregate savings, but necessarily reduces aggregate private equity, output, consumption, and labor and capital productivities.

We conclude that an increase in idiosyncratic risk (or a worsening in financial markets) shifts resources from the more productive and more risky sector (private equity) to the less productive and less risky sector (public equity), thus reducing aggregate productivity.

Note that if it were $\mu \leq 1$, public equity would dominate private equity, and therefore $\phi = K = 0$. If instead $\mu > 1$, the private-equity sector is necessarily active ($\phi, K > 0$), whereas the public-equity sector is active if and only if the risk in private equity is sufficiently high as compared to $\mu$. If instead the risk is small and $\mu$ is large, only the private-equity sector is active in equilibrium. This will become clear in the simulations of the next section.

### 6.3 Numerical Simulation

I now repeat the numerical simulations of Table 1 in the two-sector model. The new parameter that needs to be calibrated is $\mu$. For any given set of values for the other parameters ($\sigma, \gamma, \theta, \beta, \alpha, \delta$), I calibrate $\mu$ so that the implied steady-state shares of private and public equity in the aggregate capital stock are 50% each. The results are reported in Table 2.

...to be completed...

[insert Table 2 here]

I finally consider the global comparative statics of the steady state as $\sigma$ is varied continuously between 0 and 100%. Note that $\mu$ is now kept constant as $\sigma$ varies, and is calibrated so that the ratio of private to public equity is one ($X = K$) when $\sigma = 50\%$. There is some $\tilde{\sigma} \in (0, 50\%)$ such
that $X > 0$ if and only if $\sigma > \bar{\sigma}$. For $\sigma < \bar{\sigma}$, the risk in private equity is sufficiently small as compared to $\mu$ that the equilibrium is a corner solution with $X = 0$. Within this range, a local increase in $\sigma$ leads to a reduction in $K$ and $Y$, as well as in $r$. When instead $\sigma > \bar{\sigma}$, a local increase in $\sigma$ leads to a reduction in the resources allocated to private equity $(K, N)$ and an increase in the resources allocated to public equity $(X, L)$, has an ambiguous effect on aggregate savings $K + X$ but a strong negative effect on aggregate output $Y$, and increases the risk-free rate $r$.

...to be completed...

[insert Figure 1 here]

7 Pecuniary Externality and Propagation

The presence of undiversifiable entrepreneurial and capital income risk introduces a dynamic macroeconomic complementarity, as the anticipation of low income in the future leads to low risk taking and low investment in the present, which in turn implies low income in the future. Thanks to the closed-form characterization of the general equilibrium, it is easy to see exactly how this complementarity works. To simplify the exposition, consider the case that there is only private equity and ignore the equilibrium variation in $R$, $s$, or $\phi$. Conditions (11) and (12) then reduce to

$$K_{t+1} = s \phi [f(K_t) + H_t], \quad (28)$$

$$H_t = \sum_{j=1}^{\infty} R^{-j} \omega(K_{t+j}). \quad (29)$$

From (28), we see that $K_{t+1}$ increases with either $K_t$ or $H_t$. It follows that, other things equal, the path of capital $\{K_{t+1}, K_{t+2}, \ldots\}$ increases with the path of human wealth $\{H_t, H_{t+1}, \ldots\}$. This effect of future income on present investment reflects the effect of wealth on risk taking. On the other hand, (29) implies that $\{H_{t+1}, H_{t+2}, \ldots\}$ increases with $\{K_{t+1}, K_{t+2}, \ldots\}$. This feedback reflects the effect of capital on production and income. As the anticipation of high income tomorrow stimulates every agent to invest more today, and as higher aggregate investment now leads to higher aggregate capital and income tomorrow, a dynamic macroeconomic complementarity emerges, which can amplify the impact of any exogenous shock and slow down convergence to the steady state.
It is important to recognize the propagation mechanism identified above is a genuine general-equilibrium phenomenon. The complementarity derives from a pecuniary externality in risk taking: Agents do not internalize the effect that their investment choices today have on future economic activity and thereby on other’s willingness to take risk and investment today. The presence of such a pecuniary externality suggests that economic fluctuations can be inefficient, opening the door for stabilization policy intervention.

Note also that the propagation mechanism relies on two premises: First, that private investment is subject to undiversifiable idiosyncratic risk; and second, that risk taking is sensitive to anticipated future economic activity. The first premise is not satisfied in the Bewley class of models (e.g., Aiyagari, 1994, Krusell and Smith, 1998), because in these models there is no idiosyncratic risk in capital income. The second premise in not satisfied in Bernanke and Getler (1989, 1990) and Kiyotaki and Moore (1997), because agents in these models are risk neutral. It is not satisfied either in the endogenous growth models of Bencivenga and Smith (1991), Deveraux and Smith (1993), Obstfeld (1994), Krebs (2003), because in these papers own investment is the only source of income (households essentially leave in isolated islands). This explains why the propagation mechanism identified in this paper is novel to the literature.8 On the other hand, both premises are likely to hold in a large class of models beyond the one I have considered in this paper.

8 In Angeletos and Calvet (2000, 2003), current investment depends on future interest rates but not on future income, because the CARA-normal specification renders risk taking independent of wealth.

88...to be completed...

8 Concluding Remarks

This paper examined the macroeconomic effects of undiversifiable idiosyncratic investment and capital-income risk in an otherwise standard neoclassical growth economy. The incomplete-markets general equilibrium was characterized in closed form under standard assumptions for preferences and technologies. The presence of undiversifiable investment and capital-income risk was found to generate in the steady state lower levels of capital, productivity, and output as compared to complete markets. Moreover, the interaction of wealth and investment was shown to introduce a
dynamic macroeconomic complementarity, which may amplify the business cycle. The source of this complementarity was a pecuniary externality in risk taking.

When markets are complete, the optimal tax on capital is zero (Chamley, 1985; Judd, 1986; Christiano, Chari and Kehoe, 1994). Aiyagari (1995) shows that the optimal tax on capital turns positive when agents face idiosyncratic risk in their labor income. But it is an open question what are the properties of Ramsey taxation in economies with idiosyncratic risk in investment and production. Does the presence of idiosyncratic investment risk implies that we should tax capital income in order to provide partial insurance, or does the presence of pecuniary externalities imply that we should subsidize individual investment? Do the associated investment complementarities imply that there is room for stabilization policy over the business cycle? These are important questions for which the tractability of the model may permit concrete answers, thus providing more general insights about the normative implications of incomplete markets.

Other important extensions, however, will require to sacrifice tractability. For example, it would be interesting to introduce aggregate uncertainty, idiosyncratic risk in labor income, and heterogeneity in the ability or willingness to owe and run a private company. These extensions are necessary for a satisfactory quantitative assessment of the implications of entrepreneurial and investment risk for either the business cycle or the wealth distribution.

9 Appendix: Proofs

...to be completed...

References


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**Table 1**

Consider the case that private equity accounts for all capital and production. The table reports the impact of idiosyncratic investment risk on income, savings, and interest rates for a series of calibrations. The chosen parameter values are in the first six columns, and the implied effects in the last four columns. “Output loss” and “capital loss” refer to the percentage reduction in the steady-state level of output and capital as compared to complete markets. “Interest rate” is the rate of return in riskless bonds, while “private premium” is the excess return earned in private equity.
Consider the case that risk-free public equity coexists with risky private equity. The table reports the impact of idiosyncratic investment risk on income, savings, and interest rates for a series of calibrations. “Output loss” and “capital loss” now refer to the combined output and capital in private and public equity. The “interest rate” is the rate of return in either riskless bonds or public equity. The “private premium” is pinned down by the technological parameter $\mu$ and is calibrated so that private and public equity each account for half of the aggregate capital stock.

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**Table 2**
Figure 1

The plots illustrate the dependence of steady-state allocations on idiosyncratic investment risk ($\sigma$) as the latter varies from 0 to 100%. All variables except the interest rate are normalized by their complete-markets values. The benchmark calibration is used: $\gamma = 2$, $\theta = 1$, $\beta^{1-1} = 5\%$, $\alpha = 40\%$, $\delta = 10\%$, and $\mu = 3.4\%$ (that is, $\mu$ such that private and public equity each account for half the aggregate capital stock when $\sigma = 50\%$). For high values of $\sigma$, both sectors are open. For low values of $\sigma$, only private equity is active. The kinks appear exactly at the point where public equity becomes obsolete.